ELECTIVE - PAPER I

GRAPH THEORY

Objectives

To study and develop the concepts of graphs, subgraphs, lnees connectivity, Eulerian and Hamiltonian graphs, matching colorings of graphs and planar graphs.

UNIT - I

Graphs, subgraphs, Degree of a Vertex, Isomorphism of graphs, independent sets and coverings.

UNIT - II

Intersection graphs: Adjacency and incidence of matrices; Operations on graphs;

UNIT - III

Walks; trails; paths; Connectedness and components; cut point, bridge, block.

UNIT - IV

Connectivity theorems and simple problems. Eulerian graphs and Hamiltonian graphs; simple problems.

UNIT - V

Trees, theorems, and simple problems

UNIT - I

Definition : Graph (undirected Graph)

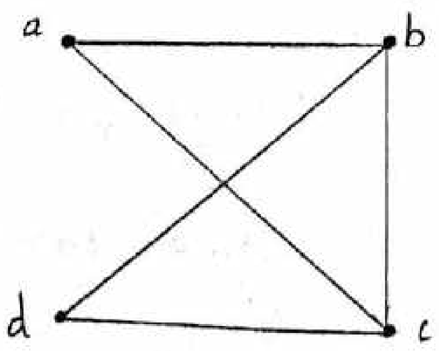
A graph G consists of a pair $(V(G), X(G))$.

Where $V(G)$ is a non-empty finite set whose elements are called points or vertices.

$X(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $X(G)$ are called lines or edges of the graph G. unordered pair

$$x = \{u, v\} \in X(G).$$

Example :

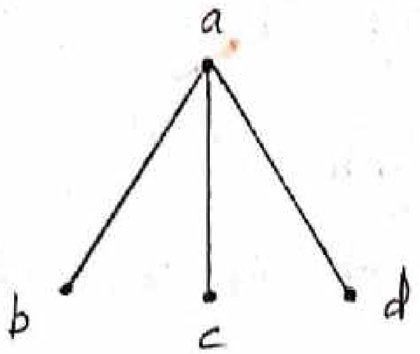
1. 

$V = \{a, b, c, d\}$.

$X = \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\}$

$G = (V, X)$ is a $(4, 5)$ graph.

2.

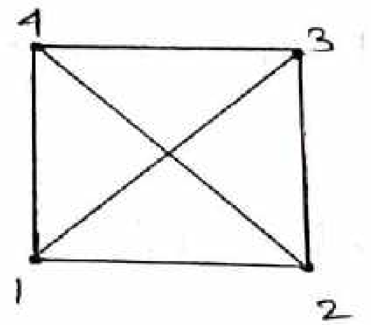


$V = \{a, b, c, d\}$

$X = \{\{a, b\}, \{a, c\}, \{a, d\}\}$

$G = (V, X)$ is a $(4, 3)$ graph.

3.



$V = \{1, 2, 3, 4\}$

$X = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 2\}, \{4, 1\}\}$

$G = (V, X)$ is a $(4, 6)$ graph.

Note:

1. A graph with p points and q lines is called a (p, q) graph

2. $V(G) = V$

   $X(G) = X$

Definition: (Adjacent or neighbors)

Two vertices u, v in an undirected graph G are called adjacent (or neighbors) in G if there is an edge x between u and v.

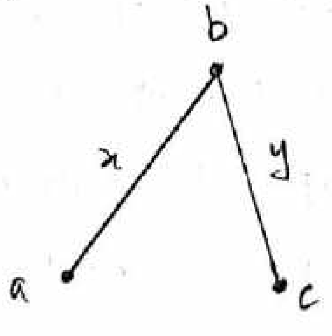
i.e) $x = uv$

If $x = uv$, we also say that the point u and the line x are incident with each other.

Definition: Adjacent lines

If two distinct lines x and y are incident with a common point then they are called adjacent lines.

Example:

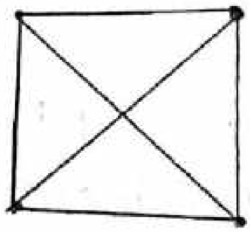


Here, the two lines x and y are incident with a common point b. So, x and y are adjacent lines.

Definition : Order and size of a graph.

The order of a graph is the number of its vertices, and its size is the number of its edges.

Example



Order = 4

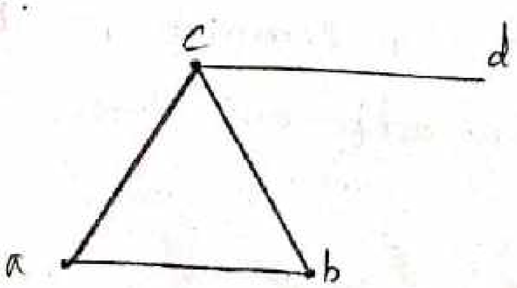
Size = 6

Definition : Neighborhood of V

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v.

If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

Example :



$N(a) = \{b, c\}$

$N(c) = \{a, b, d\}$

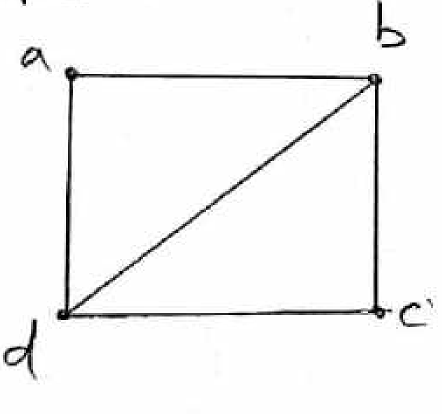
$N(b) = \{a, c\}$

$N(d) = \{c\}$

Definition: The degree of a venten

The degree of a point v_i in a graph G is the number of lines incident with v_i. The degnee of v_i is denoted by $d_G(v_i)$ on $\deg v_i$ on simply $d(v_i)$.

Example



$d(a) = 2$

$d(b) = 3$

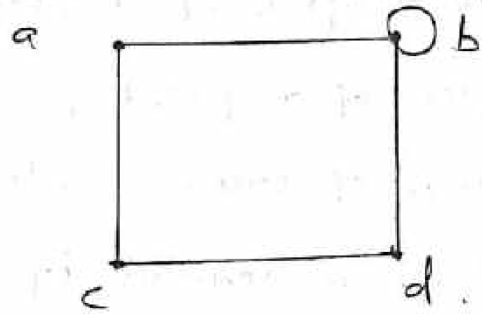
$d(c) = 2$

$d(d) = 3$

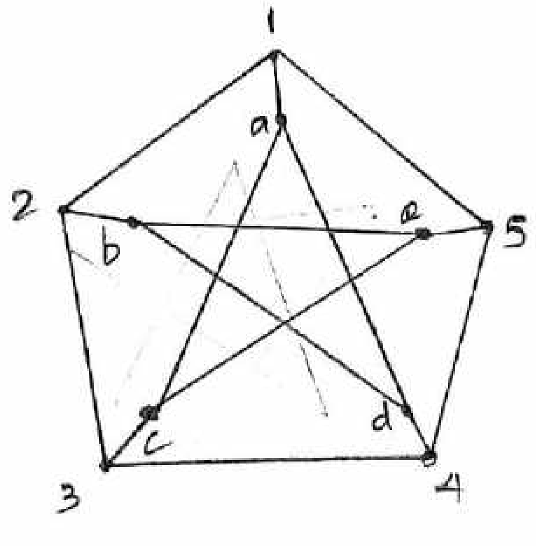
Definition : loop

A line joining a point to itself is called a loop.

Example :



Petersen graph



The (10, 15) graph is called the Petersen graph.

Definition: Multiple lines (or) Multigraphs

Multigraphs may have multiple edges connecting the same two vertices

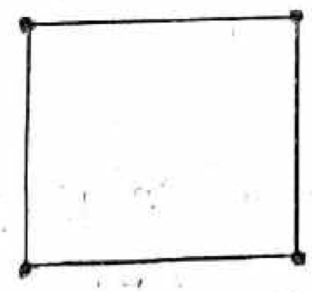
When m different edges connect the vertices u and v, we say that $\{u, v\}$ is an edge of multiplicity m.

Enample:

Definition: Simple graph

In a simple graph each edge connects two different vertices and no two edges connect the same pair of vertices

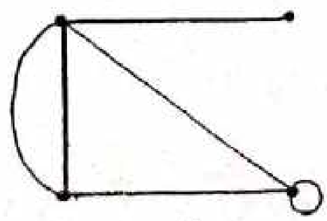
Enample:

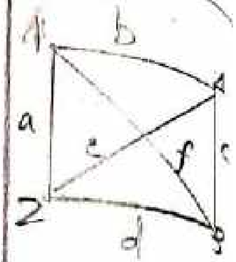
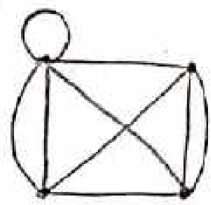


Definition: Pseudograph

A pseudograph may include loops as well as multiple edges connecting the same pair of vertices

Enample

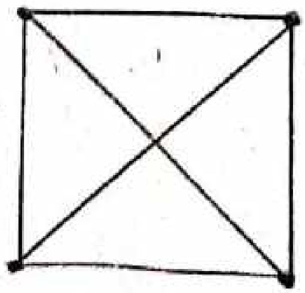


Graph	Self loop	Multiedge	Example
Simple graph	X	X	
Multigraph	X	✓	
Pseudograph	✓	✓	

Remark,

Let G be a (p, q) graph. Then $q \le \binom{p}{2}$ and $q = \binom{p}{2}$ iff any two distinct points are adjacent.

Example



Here, Here the given graph is adjacent where we take any two distinct points

$$P = 4$$

$$q \le \binom{P}{2}$$

$$\le \binom{4}{2}$$

$$\le 4C_2$$

$$q = 6$$

Definition: Complete graph

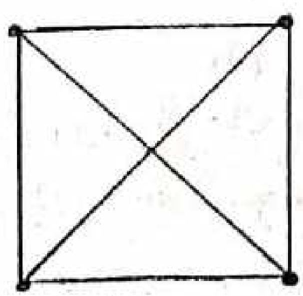
The complete graph with p points is denoted by K_p is the simple graph that contains exactly one edge between each pair of distinct vertices.

Example:

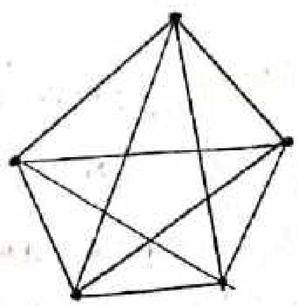
K_1

K_2

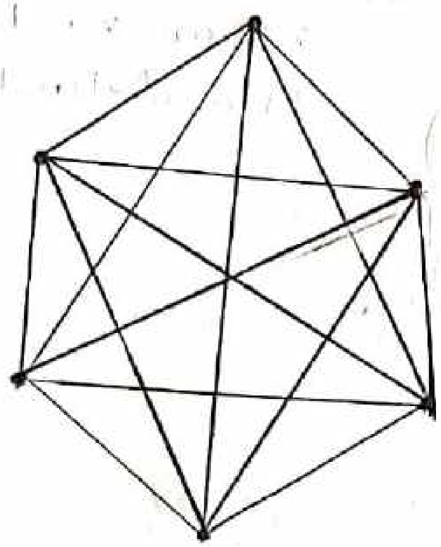
K_3



K_4



K_5



K_6

Definition: Null graph on a totally disconnected graph

A graph whose edge set is empty is called a null graph on a totally disconnected graph.

Example

1.

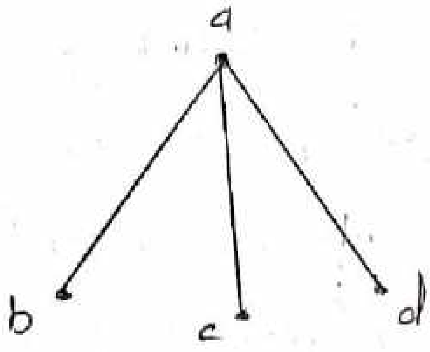
2.

Definition: labelled graph

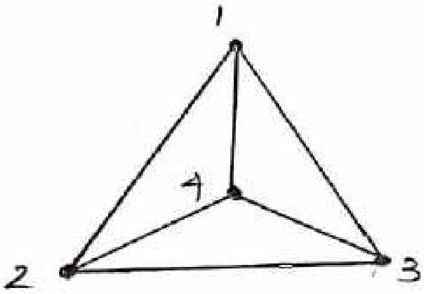
A graph G is called labelled if its p points are distinguished from one another by names such as $v_1, v_2, \ldots, v_p$.

Example:

1.



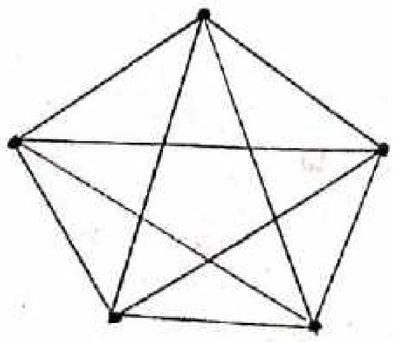
2.



Definition: Unlabelled graph

A graph G is called unlabelled if its p points are not distinguished from one another by names such as $v_1, v_2, \ldots, v_p$.

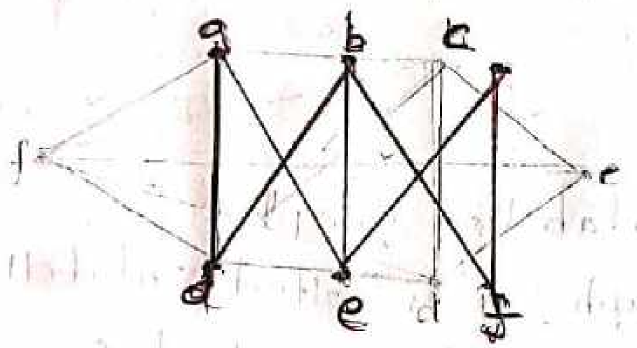
Example:



Definition: bigraph (or) bipartite graph

A graph G is called a bigraph or bipartite graph if V can be partition into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2. (V_1, V_2) is called a bipartition of G.

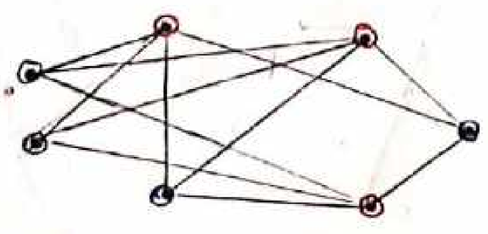
Example:



Note

An equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color.

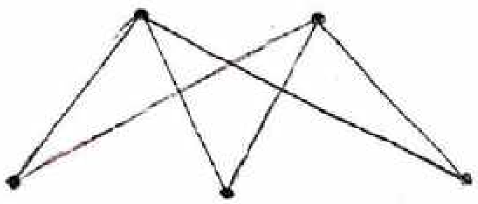
Example:



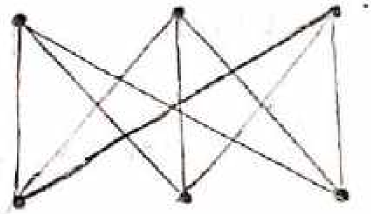
Definition: Complete bigraph

A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

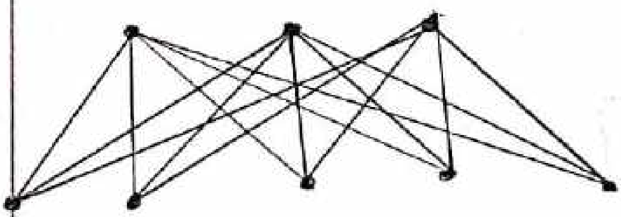
Example:



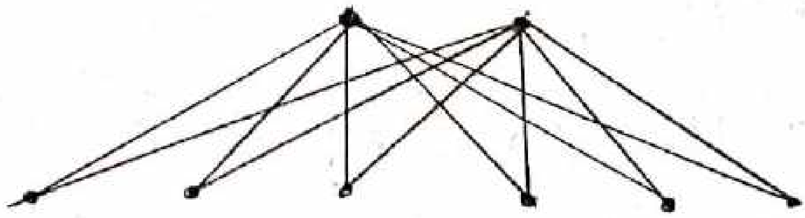
$K_{2,3}$



$K_{3,3}$



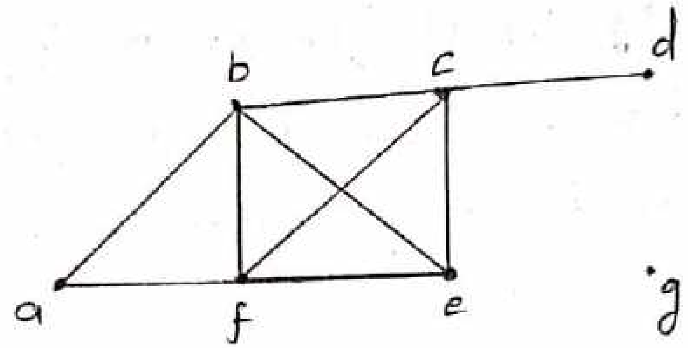
$K_{3,5}$



$K_{2,6}$.

Examples

1. What are the degrees and neighborhoods of the vertices in the graph G?



G

Solution

G: $\deg(a) = 2$

$\deg(b) = \deg(c) = \deg(f) = 4$

$\deg(d) = 1$

$\deg(e) = 3$

$\deg(g) = 0$

$N(a) = \{b, f\}$

$N(b) = \{a, c, e, f\}$

$N(c) = \{b, d, e, f\}$

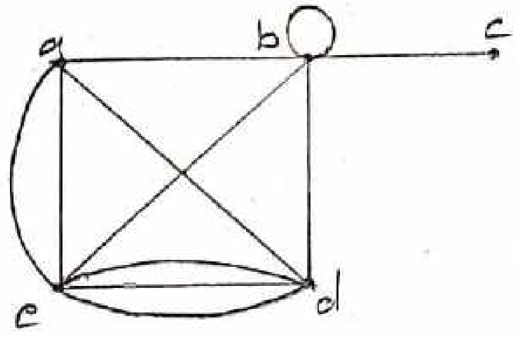
$N(d) = \{c\}$

$N(e) = \{b, c, f\}$

$N(f) = \{a, b, c, e\}$

$N(g) = \phi$

2. What are the degrees and neighborhoods of the vertices in the graph H?



Solution:

$\deg(a) = 4$

$\deg(b) = \deg(e) = 6$

$\deg(c) = 1$

$\deg(d) = 5$

$N(a) = \{b, d, e\}$

$N(b) = \{a, b, c, d, e\}$

$N(c) = \{b\}$

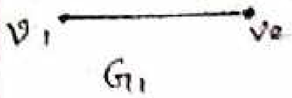
$N(d) = \{a, b, e\}$

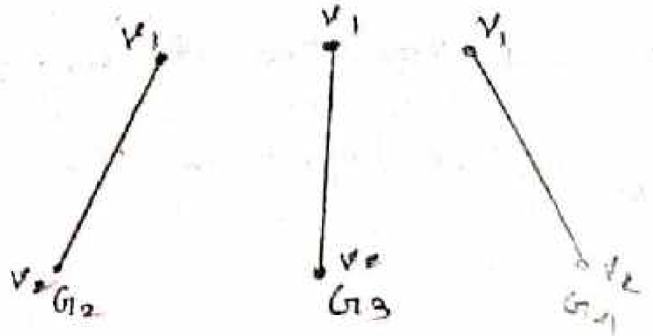
$N(e) = \{a, b, d\}$.

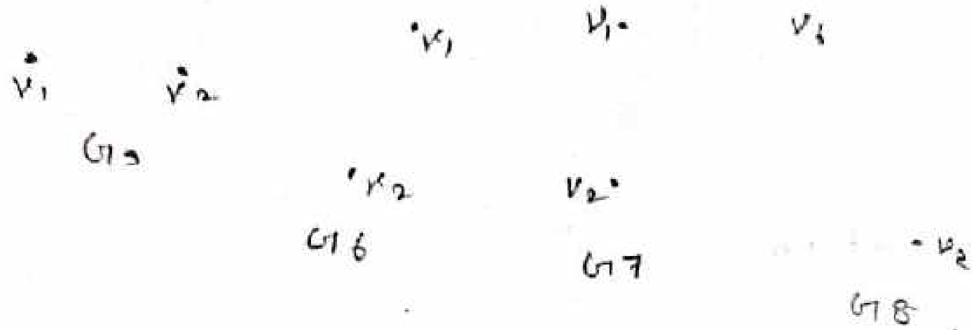
3. Draw all graphs with 1, 2, 3 and 4 points

1 points

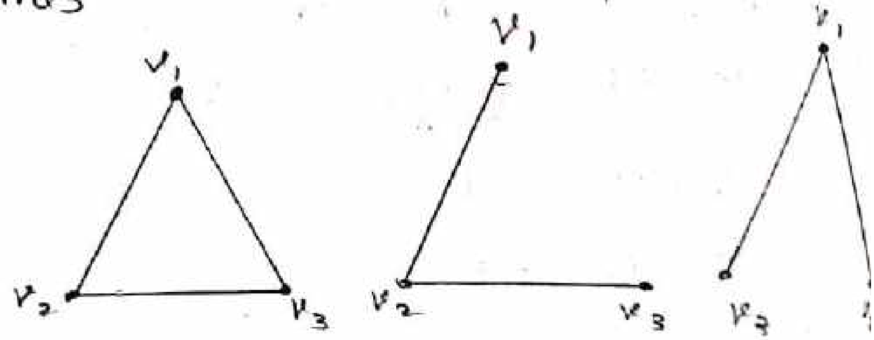
2 points



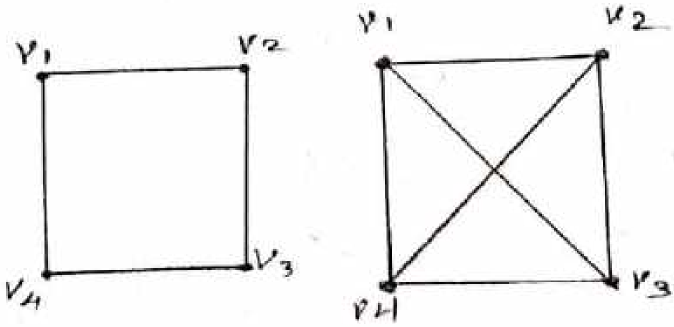


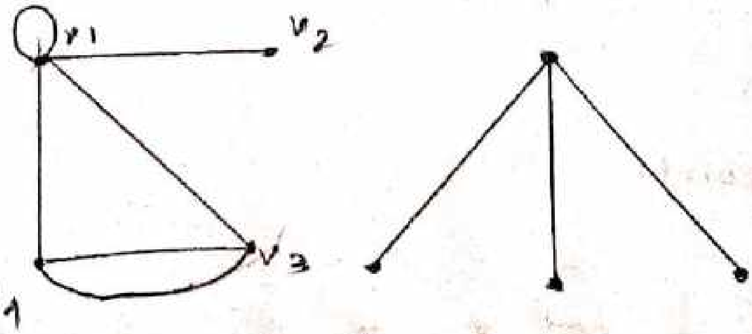


3 points



4 points





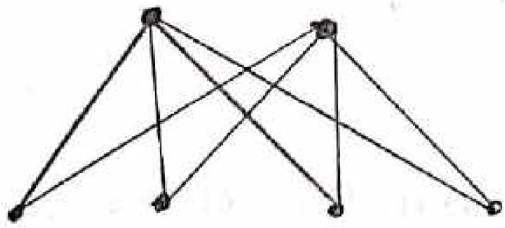
etc.

4. Find the number of points and lines in $K_{m,n}$.

$K_{m,n}$ has $m+n$ points and

$m \times n$ lines.

Example :



$K_{2,4}$

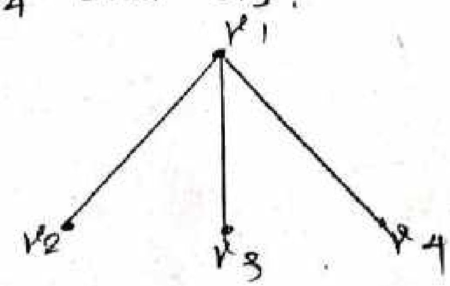
In $K_{2,4}$ there is $m+n = 2+4 = 6$

points and

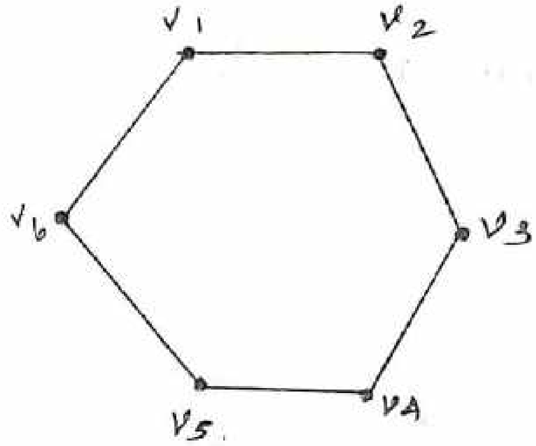
$m \times n = 2 \times 4 = 8$

lines.

3. Let $V = \{1, 2, 3, \ldots, n\}$. Let $X = \{\{i, j\} / i, j \in V$ and are relatively prime $\}$. The resulting graph (V, X) is denoted by G_n. Draw G_4 and G_5.



4. Show that C_6 is bipartite.



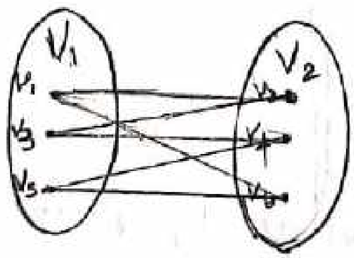
Solution:

We can partition the vertex set into

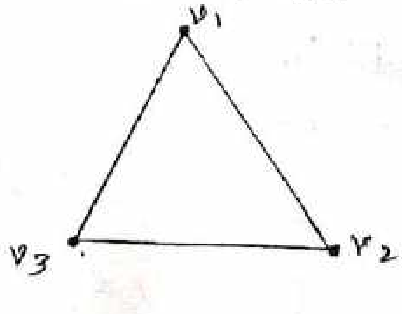
$V_1 = \{v_1, v_3, v_5\}$ and

$V_2 = \{v_2, v_4, v_6\}$.

So that every edge of C_6 connects a vertex in V_1 and V_2.



5 Show that C_3 is not bipartite.



Solution

If we divide the vertex set of C_3 into two nonempty sets, one of the two must contain two vertices. But in C_3 every vertex is connected to every other vertex. Therefore, the two vertices in the same partition are connected.

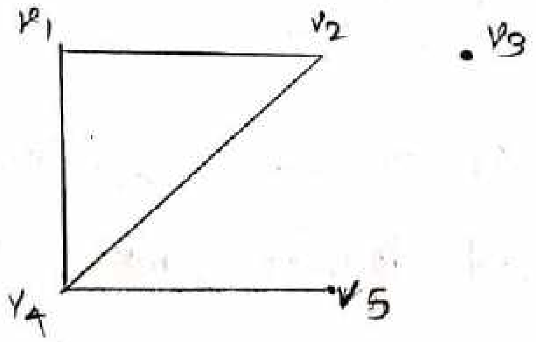
Hence C_3 is not bipartite.

Definition: Isolated point and End point

A point v of degree 0 is called an isolated point.

A point v of degree 1 is called an end point.

Example:



Here v_3 - is isolated point

v_5 - is end point.

Theorem : 1.1

The sum of the degrees of the points of a graph G is twice the number of lines.

i.e, $\sum_i \deg v_i = 2q$

Proof :

Every line of G is incident with two points. Hence everyline contributes 2 to the sum of the degrees of the points

Hence $\sum_i \deg v_i = 2q$.

Corollary :

In any graph G the number of points of odd degree is even.

Proof.

Let $v_1, v_2, \ldots, v_k$ denote the points of odd degree and

$w_1, w_2, \ldots, w_m$ denote the points of even degree in G.

By theorem 1, $\sum_{i=1}^{k} \deg v_i + \sum_{i=1}^{m} \deg w_i = 2q$

Which is even

Further $\sum_{i=1}^{m} \deg w_i$ is even

Hence $\sum_{i=1}^{k} \deg v_i$ is also even.

But $\deg v_i$ is odd for each i.

Hence k must be even.

Definition: Regular graph

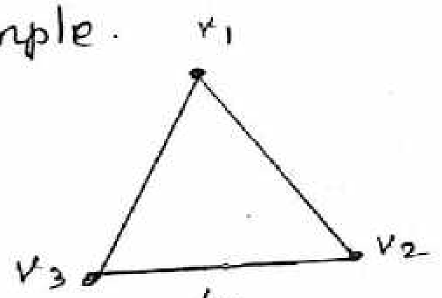
For any graph G, we define

$\delta(G) = \min\{\deg v / v \in V(G)\}$ and

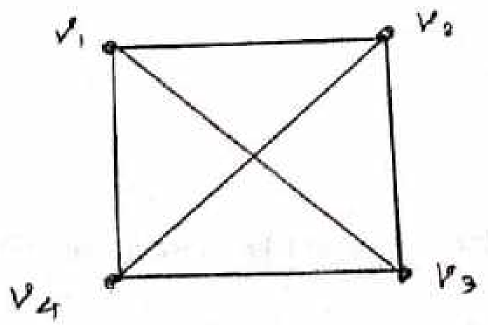
$\Delta(G) = \max\{\deg v / v \in V(G)\}$.

If all the point of G have the same degree r then $\delta(G) = \Delta(G) = r$ and in this case G is called a regular graph of degree r.

Example.



G_1



G_2

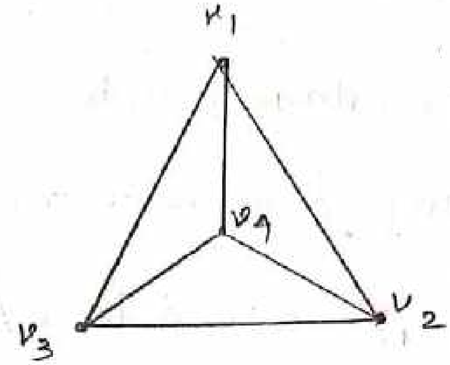
G_1 : $\delta(G) = \Delta(G) = 2$.

G_2 : $\delta(G) = \Delta(G) = 3$.

Definition: Cubic graph

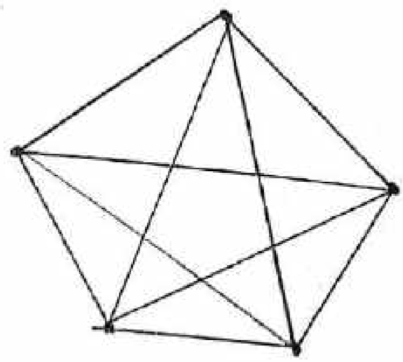
A regular graph of degree 3 is called a cubic graph.

Example.



Note

The complete graph K_p is regular of degree $p-1$.



K_5.

Theorem 1.2

Every cubic graph has an even number of points.

Proof.

Let G be a cubic graph with

p points.

Then $\sum \deg v = 3p$ which is even by theorem 1.

Hence p is even.

Problems

1. Let G be a (p, q) graph all of whose points have degree k or $k+1$. If G has $t>0$ points of degree k, show that $t = p(k+1) - 2q$.

Solution:

Since G has t points of degree k, the remaining $p-t$ points have degree $k+1$.

$$\text{Hence } \sum_{v \in V} d(v) = tk + (p-t)(k+1)$$

$$\therefore tk + (p-t)(k+1) = 2q.$$

$$\therefore t = p(k+1) - 2q.$$

2. Show that in any group of two or more people, there are always two with exactly the same number of friends inside the group.

Solution

We construct a graph G by taking the group of people as the set of points and joining two of them if they are friends.

Then $\deg v$ = number of friends of v and hence we need only to prove that atleast two points of G have the same degree.

Let $V(G) = \{v_1, v_2, \ldots v_p\}$.

Clearly $0 \leq \deg v_i \leq p-1$ for each i.

Suppose no two points of G have the same degree. Then the degrees of $v_1, v_2, \ldots, v_p$ are the integers $0, 1, 2, \ldots, p-1$ in some order.

However a point of degree $p-1$ is joined to every other point of G and hence no point can have degree zero which is a contradiction.

Hence there exist two points of G with equal degree.

3. Prove that $\delta \leq \frac{2q}{p} \leq \Delta$.

Solution.

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$.

We have $\delta \leq \deg v_i \leq \Delta$ for all i.

Hence $p\delta \leq \sum_{i=1}^{p} \deg v_i \leq p\Delta$.

$\therefore$ $p\delta \leq 2q \leq p\Delta$ (By theorem 1)

$\therefore$ $\delta \leq \frac{2q}{p} \leq \Delta$.

4. Let G be a k-regular bigraph with bipartition (V_1, V_2) and $k > 0$. Prove that $|V_1| = |V_2|$.

Solution.

Since every line of G has one end in V_1 and other end in V_2, it follow that

$$\sum_{v \in V_1} d(v) = \sum_{v \in V_2} d(v) = q$$

Also $d(v) = k$ for all $v \in V = V_1 \cup V_2$.

Hence $\sum_{v \in V_1} d(v) = k|V_1|$ and

$\sum_{v \in V_2} d(v) = k|V_2|$ so that

$$k\,|V_1| = k\,|V_2|$$

Since $k > 0$, we have $|V_1| = |V_2|$.

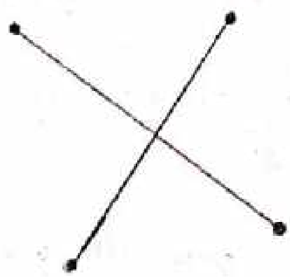
Exercises

1. Give an example of a regular graph of degree 0.

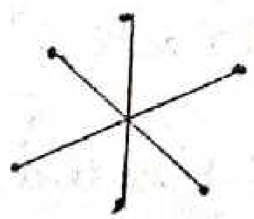
2. Give three examples for a regular graph of degree 1.

1.

2.

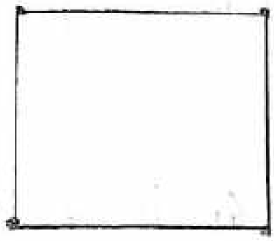


3.

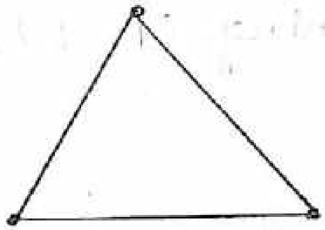


3. Give three examples for a regular graph of degree 2.

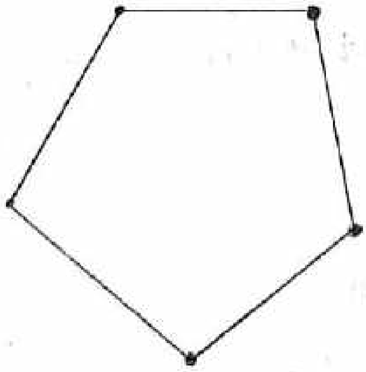
1.



2.



3.



4. What is the maximum degree of any point in a graph with p points?

Solution:

The maximum degree of any vertex in a simple graph with p vertex is p-1.

5. A (P, q) graph has t points of degree m and all other points are of degree n. Show that $(m-n)t + pn = 2q$.

Solution

Given that G is a (P, q) graph and t points of degree m.

The remaining $(p-t)$ points have degree n.

WKT, $\sum_{v \in V} d(v) = 2q$.

i.e) $mt + (p-t)n = 2q$

i.e) $(m-n)t + pn = 2q$.

Subgraphs

Definition: subgraph

A graph $H = (V_1, X_1)$ is called a subgraph of $G(V, X)$ if $V_1 \subseteq V$ and $X_1 \subseteq X$

Definitio: Supergraph

H is a subgraph of G then we say that G is a supergraph of H.

Definition: Spanning graph

H is called a spanning graph of G if $V_1 = V$.

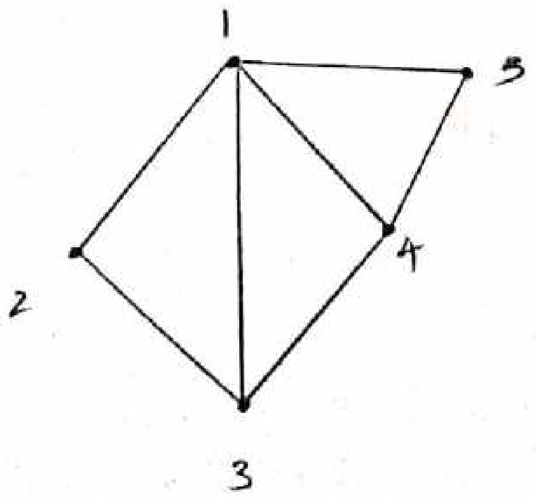
induced subgraph

H is called an induced subgraph of G if H is the maximal subgraph of G with point set V_1.

i.e) if H is an induced subgraph of G then two points are adjacent in H if and only if they are adjacent in G.

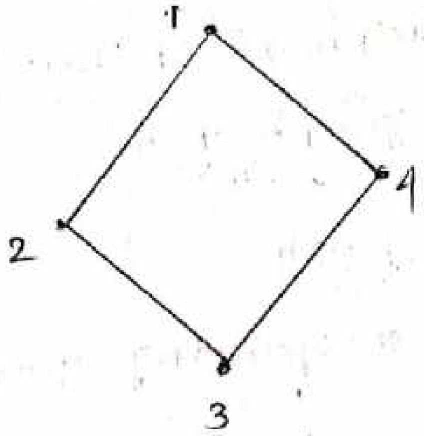
Example

1.



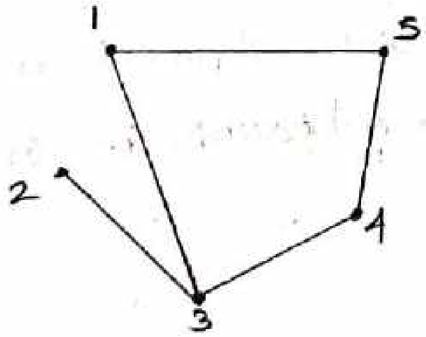
G

Subgraph H_1

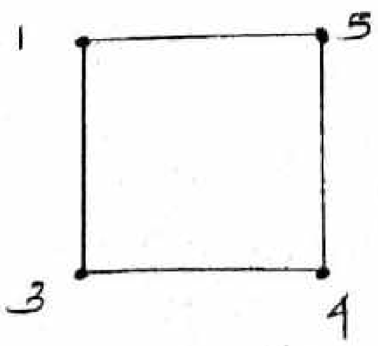


Subgrap H_2

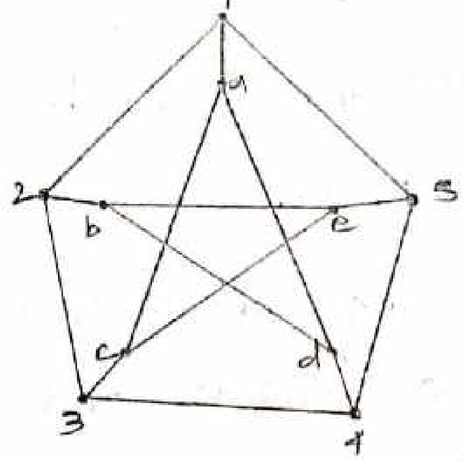
Spanning Subgraph



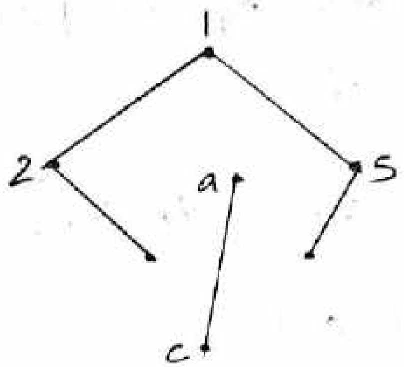
Induced Subgraph



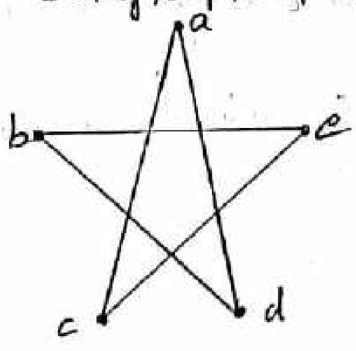
2. Consider the Petersen graph G



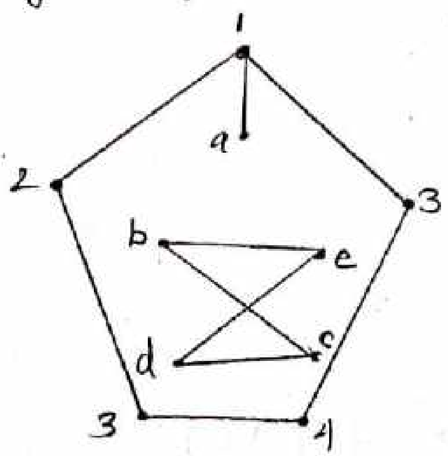
Subgraph of G



Induced subgraph of G



Spanning subgraph of G



Definition: Removal of a vertex

Let $G = (V, X)$ be a graph of

Let $v_i \in V$.

The subgraph of G obtained by removing the point v_i and all the lines incident with v_i is called the subgraph obtained by the removal of the point v_i and is denoted by $G - v_i$

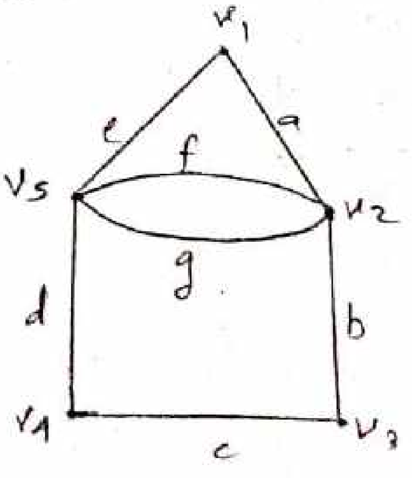
Thus if $G - v_i = (V_i, X_i)$ then

$V_i = V - \{v_i\}$ and

$X_i = \{x / x \in X$ and x is not incident

Clearly $G - v_i$ is an induced subgraph of G.

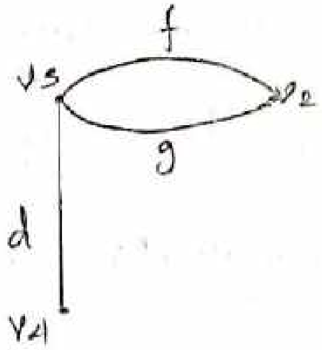
Example:



Let $V_1 = \{v_1, v_3\}$

Then $G - V_1 = G'$

G'



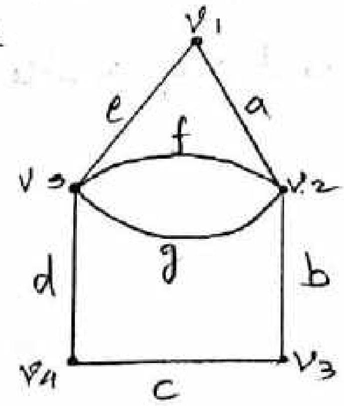
Definition : Removal of a edge.

Let $x_j \in X$.

Then $G - x_j = (V, X - \{x_j\})$ is called the subgraph of G obtained by the removal of the line x_j.

Clearly $G - x_j$ is a spanning subgraph of G which contains all the lines of G except x_j.

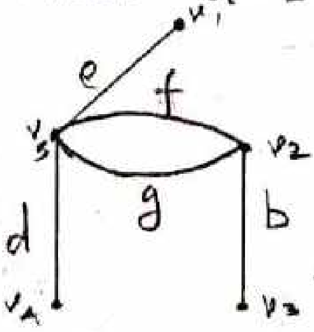
Example



Let $E_1 = \{a, c\}$

Then $G - E_1 = G'$

G'



Definition: Addition of the line

Let $G = (V, X)$ be a graph.

Let v_i, v_j be two points which are not adjacent in G.

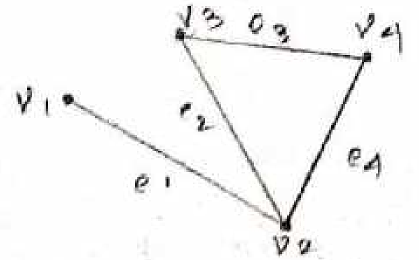
Then $G + v_i v_j = (V, X \cup \{v_i, v_j\})$ is called the graph obtained by the addition of the line $v_i v_j$ to G.

(or)

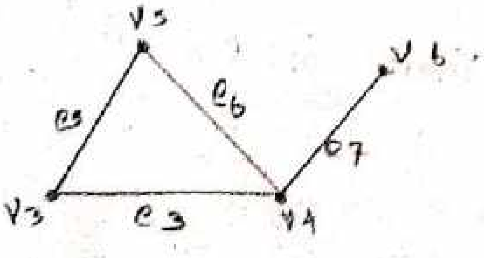
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two simple graph. The union of G_1 and G_2 is the simple graph G whose vertex set is $V = V_1 \cup V_2$ and the edge set is $E = E_1 \cup E_2$ and we write $G = G_1 \cup G_2$.

Example:

G_1 :



G_2 :



$V_1 = \{v_1, v_2, v_3, v_4\}$

$V_2 = \{v_3, v_4, v_5, v_6\}$

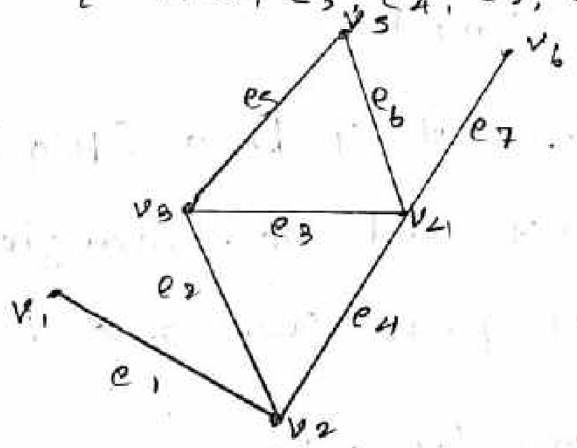
$\therefore V = V_1 \cup V_2$

$= \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

$E_1 = \{e_1, e_2, e_3, e_4\}$ and $E_2 = \{e_3, e_5, e_6, e_7\}$

$E = E_1 \cup E_2$

$= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$.



$G = G_1 \cup G_2$.

Theorem 1.3

The maximum number of lines among all p point graphs with no triangles is $\left[\frac{p^2}{4}\right]$.

($[x]$ denotes the greatest integer not exceeding the real number x).

Proof.

The result can be easily verified for $p \leq 4$.

For $p > 4$, we will prove by induction seperately for odd p and for even p.

Case i::

For odd p.

Suppose the result is true for all odd $p \leq 2n+1$.

Now let G be a (p, q) graph with $p = 2n+3$ and no triangles.

If $q = 0$, then $q \leq \left[\frac{p^2}{4}\right]$.

Hence Let $q > 0$.

Let u and v be a pair of adjacent points in G. The subgraph $G' = G - \{u, v\}$ has $2n+1$ points and no triangles.

Hence by induction hypothesis,

$$q(G') \leq \left[\frac{(2n+1)^2}{4}\right]$$

$$= \left[\frac{4n^2 + 4n + 1}{4}\right]$$

$= [n^2+n+\frac{1}{4}]$

$= n^2+n \longrightarrow (1)$

Since G has no triangles, no point of G' can be adjacent to both u and v in G. $\longrightarrow (2)$

Now, lines in G are of three types.

i) Lines of G' ($\leq n^2+n$ in number by (1))

ii) Lines between G' and $\{u, v\}$ ($\leq 2n+1$ in number by (2))

iii) Line uv.

Hence $q \leq (n^2+n)+(2n+1)+1 = n^2+3n+2$

$= \frac{1}{4}(4n^2+12n+8)$

$= \left(\frac{4n^2+12n+9}{4} - \frac{1}{4}\right)$

$= \left[\frac{(2n+3)^2}{4}\right] = \left[\frac{p^2}{4}\right]$

Also for $p = 2n+3$, the graph $K_{n+1,n+2}$ has no triangles and has $(n+1)(n+2)$ $= n^2+3n+2 = \left[\frac{p^2}{4}\right]$ lines.

Hence this maximum q is attained.

Case ii

Suppose the result is true for all even $p \leq 2n$.

Now let G be a (p, q) graph with $p = 2n+2$ and no triangles.

As before, let u and v be a pair of adjacent points in G and

Let $G' = G - \{u, v\}$.

Now G' has $2n$ points and no triangles, Hence by hypothesis

$$q(G') \leq \left[\frac{(2n)^2}{4}\right] = n^2 \longrightarrow (3)$$

Lines in G are of three types.

(i) Lines of G' ($\leq n^2$ in number by (3))

ii) Lines between G' and $\{u, v\}$ ($\leq 2n$ in number by an argument similar to (2))

iii) line uv.

Hence $q \leq n^2 + 2n + 1 = (n+1)^2 = \frac{(2n+2)^2}{4}$

$$= \left[\frac{p^2}{4}\right]$$

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Hence the result holds for even p also.

We see that for $p = 2n+2$.

$K_{n+1, n+1}$ is a $(p, [\frac{p^2}{4}])$ graph without triangles.

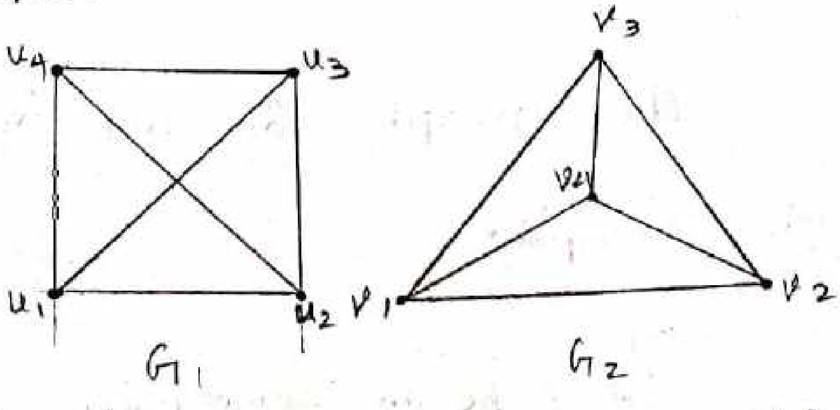
Definition : Isomorphism

Two groups $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ are said to be isomorphic if there exists a bijection $f : V_1 \rightarrow V_2$ such that $u, v \in V_1$ are adjacent in G_1 if and only if $f(u), f(v) \in V_2$ are adjacent in G_2.

If G_1 is isomorphic to G_2 then we write $G_1 \cong G_2$. The map f is called an isomorphism from G_1 to G_2.

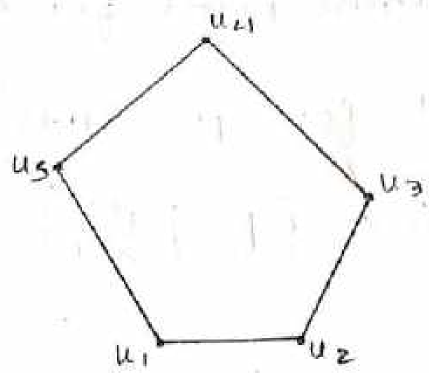
Example:

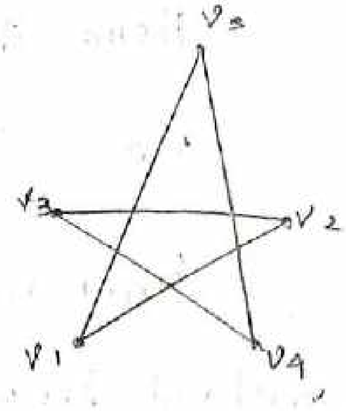
1.



$f(u_i) = v_i$ is an isomorphism between two groups G_1 and G_2 for $i = 1, 2, 3, 4$.

2.



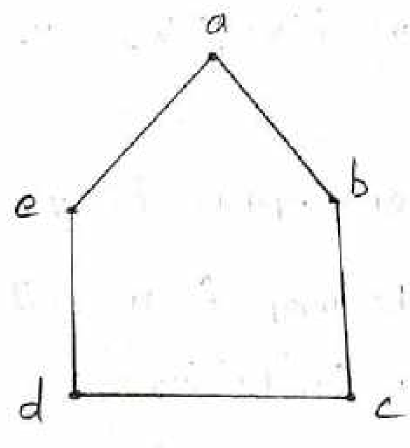


$f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_3$,

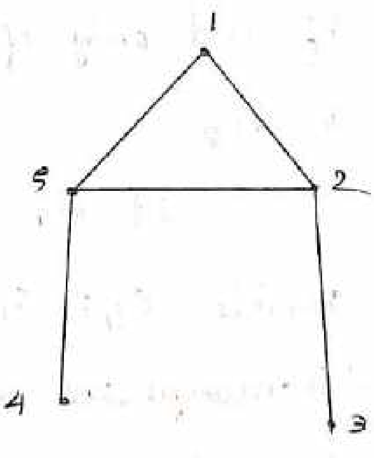
$f(u_4) = v_4$, $f(u_5) = v_5$.

i.e) $f(u_i) = v_i$ is an isomorphism between two groups G_1 and G_2 for $i = 1, 2, \cdots, 5$

3.



G_1



G_2

The graph G_1 and G_2 are not isomorphic.

Theorem 1.4

Let f be an isomorphism of the group $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$. Let $v \in V_1$.

Then $\deg v = \deg f(v)$

i.e). isomorphism preserves the degree of vertices.

Proof.

Let f be an isomorphism of the group $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$.

Given that $v \in V_1$.

$\therefore$ a point $u \in V_1$ is adjacent to v in G_1 if and only if $f(u)$ is adjacent to $f(v)$ in G_2.

Also, f is a bijection

Hence the number of points in V_1 which are adjacent to v is equal to the number of points in V_2 which are adjacent to $f(v)$

$\therefore \deg v = \deg f(v)$.

Hence isomorphism preserves the degree of vertices.

Remark.

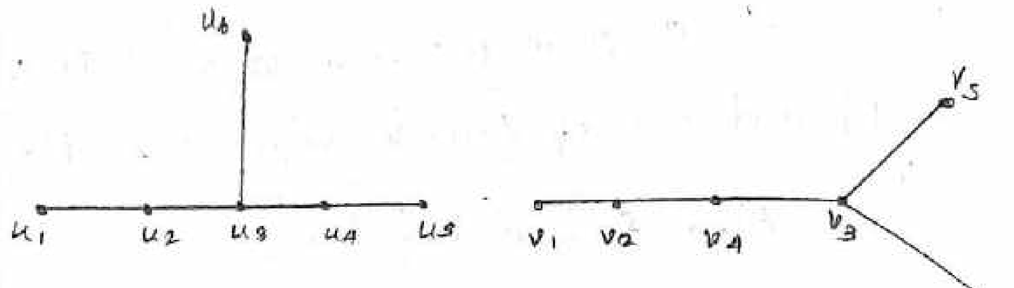
1. Two isomorphic group have the same number of points and the same number of lines.

2. The converse of the above theorem is not true.

i.e. If the degrees of the vertices of two graphs are equal then the two graphs need not be isomorphic.

Example:

Consider the two graphs G_1, G_2



G_1 $\qquad$ G_2.

Here, $\deg u_i = \deg v_i$ for $i = 1, 2, 3, 4, 5, 6$.

But G_1 and G_2 are not isomorphic, because u_2 is adjacent to u_3 in G_1 but v_2 is not adjacent to v_3 in G_2.

Definition: Automorphism

An isomorphism of a group onto itself is called an automorphism of G.

Remark.

The set of all automorphism of G is a group. This group is denoted by $\Gamma(G)$ and is called the automorphism group of G.

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Definition: Complement graph.

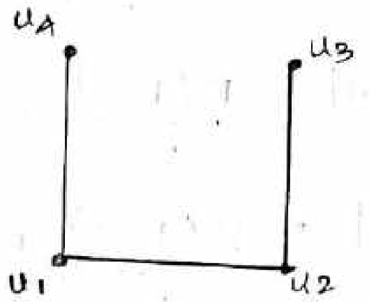
Let $G = (V, X)$ be a group. the complement $\bar{G}$ of G is defined to be the graph which has V as its set of points and two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

Definition: Self Complementary

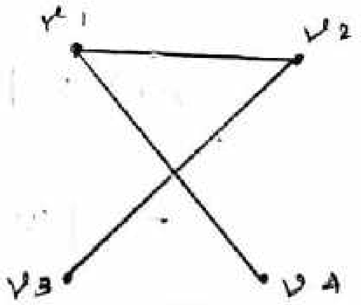
The graph G is said to be a self complementary graph if G is isomorphic to $\bar{G}$.

Example:

1.

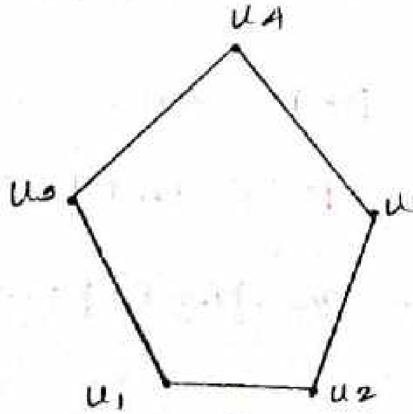


G

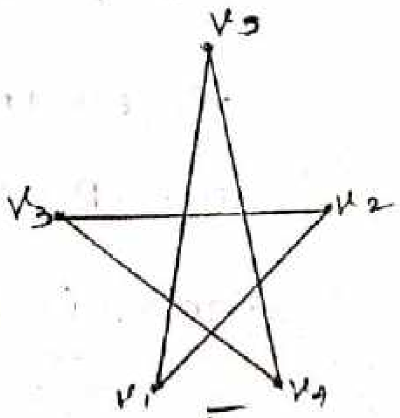


$\bar{G}$

2.



G



$\bar{G}$

Problems

1. Prove that any self complementary graph has 4 n or 4n+1 points.

Solution:

Let G be a complementary graph with p points.

$\therefore G \cong \overline{G}$

$\Rightarrow |X(G)| = |X(\overline{G})|$.

Also, $|X(G)| + |X(\overline{G})|$ = Number of edges in K_p

$= \binom{p}{2} = \frac{p(p-1)}{2}$

i.e) $2|X(G)| = \frac{p(p-1)}{2}$.

$\Rightarrow |X(G)| = \frac{p(p-1)}{4}$ is an integer.

i.e) $p(p-1) = 4n$, where $n \in Z$.

i.e) p or $(p-1)$ is a multiple of 4.

$\Rightarrow p = 4n$ or $p-1 = 4n$.

i.e. $p = 4n$ or $p = 4n+1$.

Hence, G has $4n$ or $4n+1$ points.

2. Prove that $\Gamma(G) = \Gamma(\bar{G})$.

Solution.

We know that, $\Gamma(G)$ is a group automorphism of G.

First, we prove that, $\Gamma(G) \subseteq \Gamma(\bar{G})$

Let $f \in \Gamma(G) \Rightarrow f : G \rightarrow G$ is an isomorphism

Let $u, v \in V(G)$.

Now, u, v are adjacent in $\bar{G} \Leftrightarrow u, v$ are not adjacent in G.

$\Leftrightarrow f(u), f(v)$ are not adjacent in G, since f is an automorphism of G.

$\Leftrightarrow f(u), f(v)$ are adjacent in $\bar{G}$.

$\therefore f : \bar{G} \rightarrow \bar{G}$ is an automorphism.

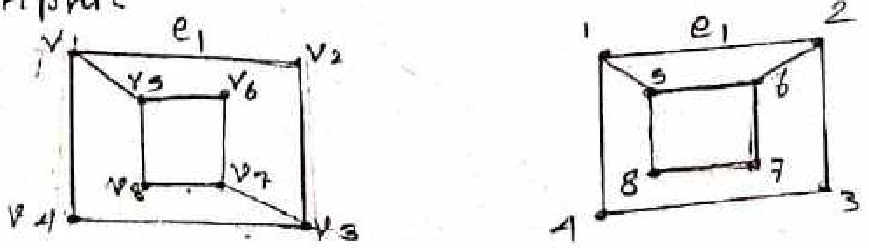
$\therefore f \in \Gamma(\bar{G})$.

Hence, $\Gamma(G) \subseteq \Gamma(\bar{G})$.

Similarly, we can prove that $\Gamma(\bar{G}) \subseteq \Gamma(G)$.

$\therefore \Gamma(G) = \Gamma(\bar{G})$.

3. Show that the two graphs given below are not isomorphic



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$f : V_1 \to V_2$ $\qquad$ $g : E_1 \to E_2$

$f(v_1) = 1$ $\qquad$ $g(e_1) = g(v_1 v_2)$

$f(v_2) = 3$ $\qquad$ $= 12$.

$f(v_3) = 2$

$f(v_4) = 4$

$f(v_5) = 5$

$f(v_6) = 7$

$f(v_7) = 6$

$f(v_8) = 8$.

Independent sets and coverings.

Definition: Independent set

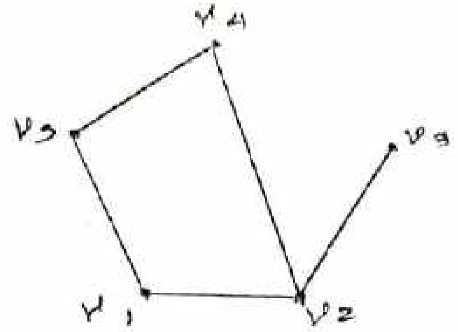
Let $G = (V, X)$ be a graph. A subset S of V is called an independent set of G if no two vertices of S are adjacent in G.

An independent set S is said to be maximum if G has no independent set S' with $|S'| > |S|$.

The number of vertices in a maximum independent set is called the independent number of G and is denoted by α.

Example

Consider the graph given.



$S_1 = \{v_1, v_3, v_4\}$

$S_2 = \{v_2, v_5\}$

$S_3 = \{v_3, v_4\}$ are independent sets.

S_1 is the maximal independent set.

$\therefore \alpha = |S_1| = 3$.

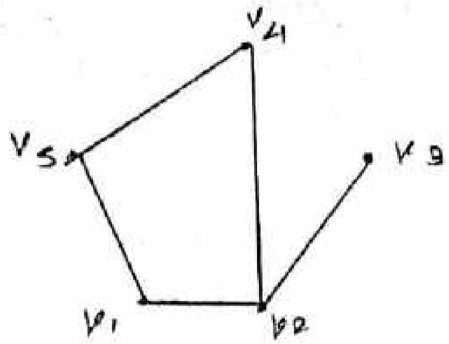
Definition: Covering

A covering of a graph $G = (V, X)$ is a subset K of V such that every line of G is incident with the vertex in K.

A covering K is called a minimum covering if G has no covering K' with $|K'| < |K|$.

The number of vertices in a minimum covering of G is called the covering number of g and is denoted by β.

Example.



$k_1 = \{v_1, v_3, v_4\}$

$k_2 = \{v_2, v_5\}$

$k_3 = \{v_4, v_2, v_5\}$ are covering of G.

k_2 is the minimum covering

$\beta = |k_2| = 2$.

Theorem 1.5

A set $S \subseteq V$ is an independent set G if and only if $V-S$ is a covering of S.

Proof.

Let $G = (V, X)$ be a graph.

By definition, A set $S \subseteq V$ is an independent iff no two vertices of S are adjacent.

i.e. iff every line of S incident with atleast one point $V-S$.

i.e. iff $V-S$ is a covering of G.

Corrollary. For any graph G of p vertices
$$\alpha + \beta = p$$

Proof.

Let G be a group with p vertices.

Let S be a maximum independent set and K be a minimum covering of G.

$\therefore |S| = \alpha$ and $|K| = \beta$.

By theorem 5, S is an independent set iff $V-S$ is an covering.

But, K is a minimum covering of G.

Hence, $|K| \leq |V-S|$

i.e. $\beta \leq p - \alpha$

i.e. $\beta + \alpha \leq p \longrightarrow (1)$

Also K is a covering iff $V-K$ is an independent set.

But, S is a maximum independent set.

$\therefore |V-K| \leq |S|$.

i.e. $p - \beta \leq \alpha$

i.e. $p \leq \alpha + \beta \longrightarrow (2)$

From equations (1) and (2), we get $\alpha + \beta = p$.

Definition: Line Covering

A line covering of graph $G = (V, X)$ is a subset L of X such that every vertex is incident with a line of L.

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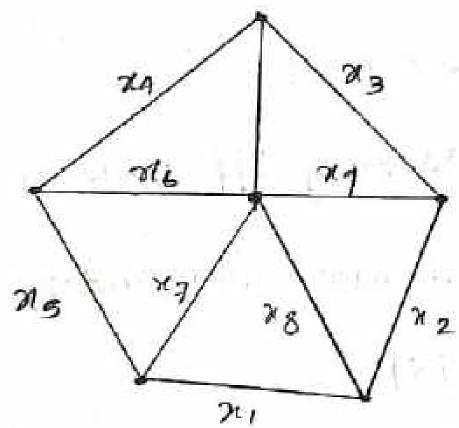
The number of lines in a minimum line covering of G is called the line covering number of G and is denoted by β'

A line set is called independent if no two of them are adjacent.

The number of lines in a maximum independent set of lines is called the edge independent number and is denoted by α'.

Example:



$L_1 = \{x_1, x_3, x_6\}$

$L_2 = \{x_2, x_4\}$

$L_3 = \{x_3, x_5, x_8\}$ are edge independent sets

Here, L_1 and L_3 are maximum edge independent sets.

$$\therefore \alpha' = |L_1| = |L_3| = 3.$$

$K_1 = \{x_1, x_3, x_6\}$ and

$K_2 = \{x_3, x_5, x_8\}$ are edge covering sets.

i.e. K_1 and K_2 are minimum edge covering sets.

$$\therefore \beta' = |K_1| = |K_2| = 3.$$

Gallia's Theorem

Theorem 1.6

For any non-trivial graph with p vertices, $\alpha' + \beta' = p$.

Proof.

Let G be a (p, q) graph.

WKT, α' is the maximal edge independent set and β' is the minimum edge covering number.

Let S be the maximum independent set of lines of G.

$$\therefore |S| = \alpha'.$$

Let M be a set of lines, one incident line for each of the $p - 2\alpha'$ points of G not covered by any line of S.

Clearly, $S \cup M$ is a line covering of G.

$$\therefore |S \cup M| \geq \beta'$$

i.e. $\alpha' + p - 2\alpha' \geq \beta'$

i.e. $p \geq \alpha' + \beta'$ ——— (1)

Let T be the minimum edge covering set.

$$\therefore |T| = \beta'.$$

T cannot have a line x, both of whose ends are also incident with lines of T other than x.

Hence $G[T]$, the spanning subgraph of G induced by T is the union of stars.

Hence each line of T is incident with atleast one end point of $G[T]$.

Let W be the set of end points of $G[T]$ consisting of exactly one end point for each line of T.

$$\therefore |W| = |T| = \beta'$$

Hence, $p = |T| +$ number of stars in $G[T]$

i.e. $p = \beta' +$ number of stars in $G[T]$ $\longrightarrow (2)$

By choosing one line from each star, we get a set of independent lines.

$\therefore \alpha' \geq$ (number of stars in $G[T]$)

i.e. $p \leq \beta' + \alpha'$ $\longrightarrow (3)$

Hence, from equations (1) and (3), $\alpha' + \beta' = p$.

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Unit - II

Intersection Graphs and Line Graphs.

Definition : Intersection Graph

Let $F = \{S_1, S_2, \ldots, S_p\}$ be a non-empty family of distinct non-empty subsets of a given set S. The intersection graph of F, denoted by $\Omega(F)$ is defined as follows.

The set of points V of $\Omega(F)$ is F itself and two points S_i, S_j are adjacent if $i \neq j$ and $S_i \cap S_j \neq \phi$.

A graph G is called an intersection graph on S if there exists a family F of subsets of S such that G is isomorphic to $\Omega(F)$.

Example:

$A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

$C = \{1, 5, 6\}$

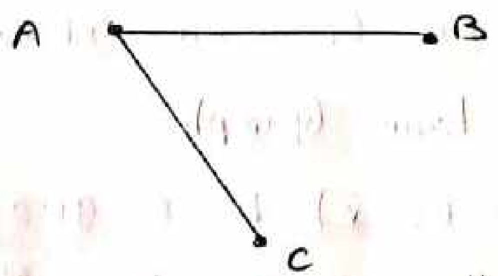
$A \cap B = \{2, 3\}$; $A \cap C = \{1\}$, $B \cap C = \phi$.

nc_2

$= \frac{n(n-1)}{2}$

$= \frac{3 \times 2}{2}$

$= \frac{6}{2} = 3$



Theorem. 2.1

Every graph is an intersection graph

Proof.

Let $G = (V, X)$ be a graph.

Let $S = V \cup X$.

For each $v_i \in V$,

let $S_i = \{v_i\} \cup \{x \in X / v_i \in x\}$.

Clearly

$F = \{S_1, S_2, \ldots, S_p\}$ is a family of distinct non-empty subsets of S.

Further if v_i, v_j are adjacent in V then $v_i v_j \in S_i \cap S_j$ and hence $S_i \cap S_j \neq \phi$.

Conversely if $S_i \cap S_j \neq \phi$ then the element common to $S_i \cap S_j$ is the line joining v_i and v_j so that v_i, v_j are adjacent in G.

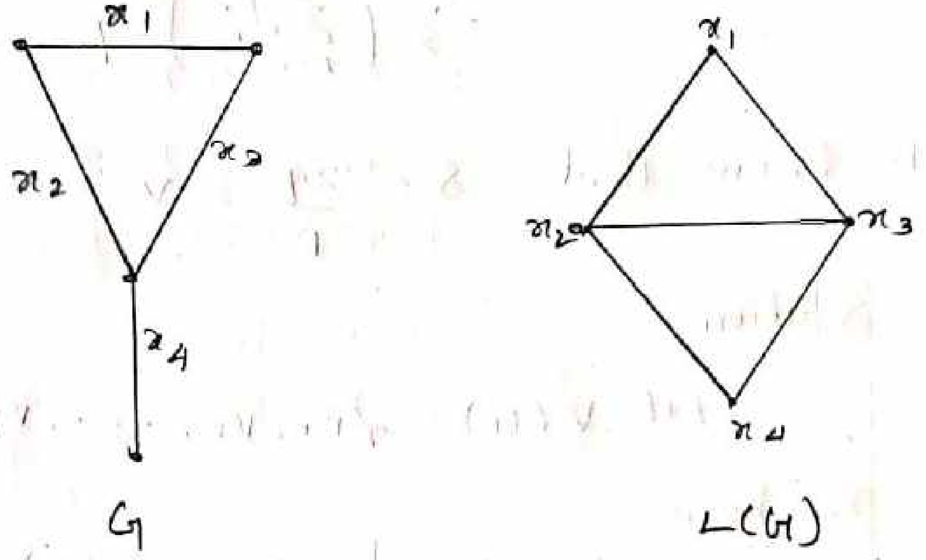
Thus $f : V \rightarrow F$ defined by $f(v_i) = S_i$ is an isomorphism of G to $\Omega(F)$.

Hence G is an intersection graph.

Definition: Line Graph

Let $G = (V, X)$ be a graph with $X \neq \phi$. Then X can be thought of as a family of 2 element subsets of V. The intersection graph $\Omega(X)$ is called the line graph of G and is denoted by L(G). Thus the points L(G) are the lines of G and two points in L(G) are adjacent iff the corresponding lines are adjacent in G.

Example: 2.2



Theorem 2.2

Let G be a (p, q) graph. Then L(G) is a (q, q_L) graph where $q_L = \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q$.

Proof

By defn, number of points in L(G) is q.

To find the number of lines in L(G). Any two of the d_i lines incident with v_i are adjacent in L(G) and hence we get $\frac{d_i(d_i-1)}{2}$ lines in L(G).

Hence $q_L = \sum_{i=1}^{p} \frac{d_i(d_i-1)}{2}$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}\left(\sum_{i=1}^{p} d_i\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}(2q)$$

(by theorem 1.1)

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q$$

1. Prove that $\delta \leq \frac{2q}{p} \leq \Delta$

Solution.

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$

We have.

$$\delta(G) \leq \deg v_i \leq \Delta(G)$$

$$p\delta(G) \leq \sum \deg v_i \leq p\,\Delta(G)$$

$$p\delta \leq 2q \leq \Delta p.$$

$$\delta \leq \frac{2q}{p} \leq \Delta.$$

Definition: Matrices.

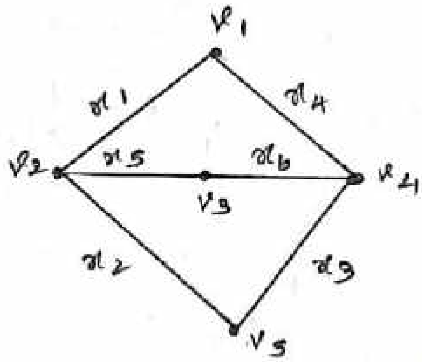
Let $G = (V, X)$ be a (p, q) graph. Let $V = \{v_1, v_2, \ldots, v_p\}$. The $p \times p$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

is called the adjacency matrix of the graph G.

Example:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$



Definition: Incidence Matrix

Let $G = (V, X)$ be a (p, q) graph.

Let $V = \{v_1, v_2, \ldots, v_p\}$ and

$X = \{x_1, x_2, \ldots, x_q\}$.

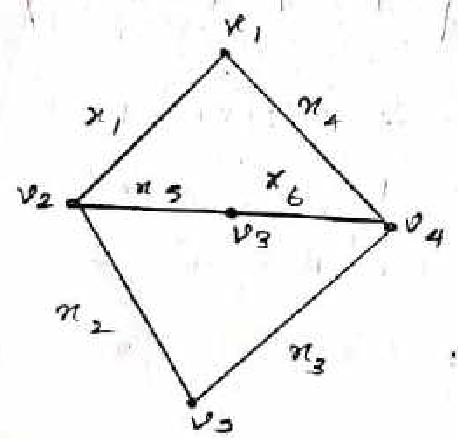
The $p \times q$ matrix $B = (b_{ij})$ where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } x_j \\ 0 & \text{otherwise.} \end{cases}$$

is called the incidence matrix of the graph.

Example.

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$



Note:

We make the following observations regarding the adjacency and incidence matrices of a graph.

1. The adjacency matrix A is symmetric.

2. The sum of i^{th} row of A is equal to the degree of v_i

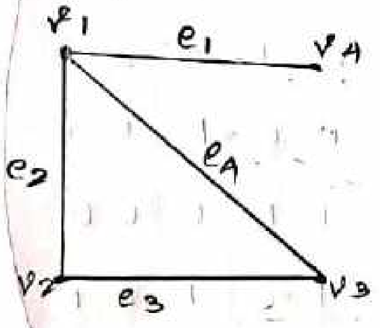
3. The entries along the principal diagonal of A are zero.

4. Since each edge is incident with exactly two vertices, each column of the incidence matrix B contains exactly two 1's.

5. The sum of i^{th} row of B is equal to the degree of v_i.

Example

1. Write the incidence and adjacency matrix for below graph.

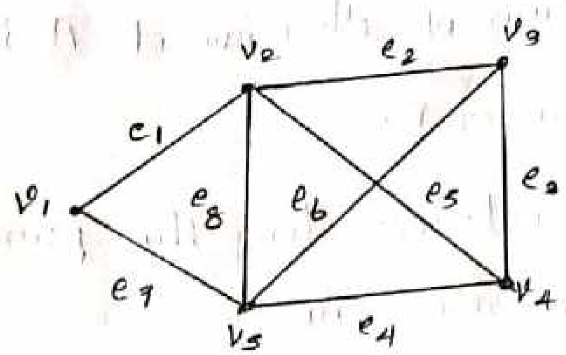


Solution

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}_{4\times 4}$$

$$B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}_{4\times 4}$$

2.



Solution

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}_{5 \times 5}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Operations on Graphs

Definition

Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two graphs with $V_1 \cap V_2 = \phi$. We define.

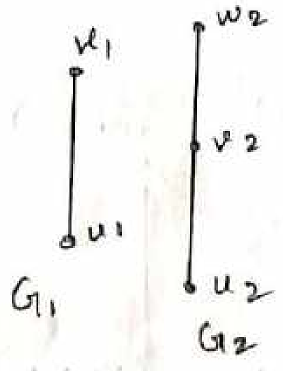
(i) The union $G_1 \cup G_2$ to be (V, X) where $V = V_1 \cup V_2$ and $X = X_1 \cup X_2$

ii) The sum $G_1 + G_2$ as $G_1 \cup G_2$ together with all the lines joining points of V_1 to points of V_2.

iii) The product $G_1 \times G_2$ as having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.

iv) The composition $G_1[G_2]$ as having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent to v_1 in G_1 or ($u_1 = v_1$ and u_2 is adjacent to v_2 in G_2).

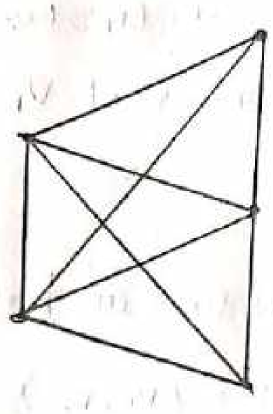
Example



Union

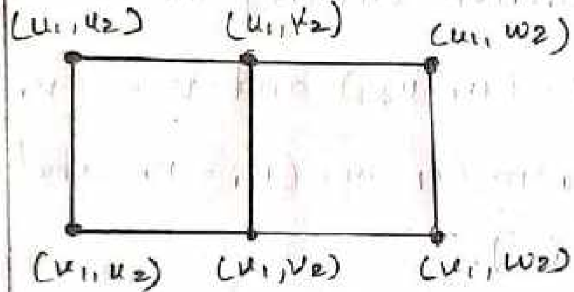
$G_1 \cup G_2$

Sum



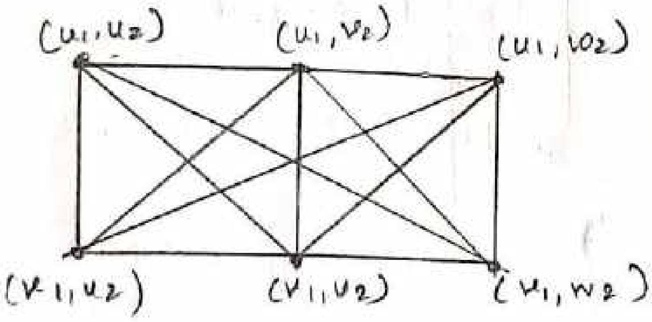
$G_1 + G_2$

Product



$G_1 \times G_2$

Composition



$G_1 [G_2]$

We note that $\overline{K_m} + \overline{K_n} = K_{m,n}$.

Theorem 2.3

Let G_1 be a (p_1, q_1) graph and G_2 a (p_2, q_2) graph.

(i) $G_1 \cup G_2$ is a (p_1+p_2, q_1+q_2) graph.

(ii) $G_1 + G_2$ is a $(p_1+p_2, q_1+q_2+p_1p_2)$ graph

(iii) $G_1 \times G_2$ is a $(p_1p_2, q_1p_2+q_2p_1)$ graph

iv) $G_1[G_2]$ is a $(p_1p_2, p_1q_2+p_2^2q_1)$ graph.

Proof.

(i) is obvious.

(ii) number of lines in $G_1 + G_2$

= number of lines in G_1 + number of lines in G_2 + number of lines joining points of V_1 to points of V_2.

$= q_1 + q_2 + p_1p_2$.

Hence we get (ii).

(iii) Clearly number of points in $G_1 \times G_2$ is p_1p_2.

Now, let $(u_1, u_2) \in V_1 \times V_2$.

The points adjacent to (u_1, u_2) are (u_1, v_2) where u_2 is adjacent to v_2 and (v_1, u_2) where v_1 is adjacent to u_1.

$\therefore \deg(u_1, u_2) = \deg u_1 + \deg u_2$

The total no of lines in $G_1 \times G_2$

$= \frac{1}{2}[\sum_{i,j} \deg(u_i) + \deg(v_j)]$

[by theorem 1.1]

$$= \frac{1}{2} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (\deg u_i + \deg v_j) \text{ where } u_i \in V_1,\ v_j \in V_2$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} \left(p_2 \deg u_i + \sum_{j=1}^{p_2} \deg v_j\right)$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} (p_2 \deg u_i + 2q_2)$$

$$= \frac{1}{2} (p_2 2q_1 + p_1 2q_2)$$

$$= p_2 q_1 + p_1 q_2.$$

Thus $G_1 \times G_2$ is a $(p_1 p_2,\ p_2 q_1 + p_1 q_2)$ graph

(v). $G_1[G_2]$ is the graph with vertex set $V_1 \times V_2$

$$\therefore |V_1 \times V_2| = p_1 p_2.$$

Also, we know that, the points (u_1, u_2) and (v_1, v_2) are adjacent in $G_1[G_2]$ if u_1 is adjacent to v_1 in G_1 or ($u_1 = v_1$ and u_2 is adjacent to v_2 in G_2).

$$\therefore \deg(u_1, u_2) = \deg u_2 + p_2 \deg u_1$$

$$= p_2 \deg u_1 + \deg u_2$$

The total number of lines in $G_1[G_2]$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} deg(u_i, u_j)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} (p_2 deg u_i + deg u_j)\right]$$

$$= \frac{1}{2}\left[p_2 \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} (deg u_i)\right] + \frac{1}{2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} (deg u_j)\right]$$

$$= \frac{1}{2}\left[p_2^2 2q_1 + p_1 2q_2\right]$$

$$= p_2^2 q_1 + p_1 q_2.$$

Hence $G_1[G_2]$ is a $(p_1p_2, p_1q_2 + p_2^2 q_1)$ graph.

Unit - III

WALKS, TRIALS AND PATHS

Definition: Walk

A walk of a graph G is defined as a finite alternating sequence of points and lines of the form $v_0, x_1, v_1, x_2, v_2, x_2, x_3, \ldots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_i.

Definition: Initial and terminal point, length of the walk.

The walk joins v_0 and v_n is called $v_0 - v_n$ walk. v_0 is called the initial point and v_n is called the terminal point of the walk. The number of lines in the walk is called the length of the walk.

Note

1. No edge appears more than once in a walk.

2. A vertex may appear more than once.

3. The $v_0 - v_n$ walk is also denoted by $v_0, v_1, \ldots, v_n$.

4. A single point is considered as a walk of length 0.

Definition: Closed walk.

A walk which begins and ends at the same point is called a closed walk

i-e) a $v_0 - v_n$ walk is called walk if $v_0 = v_n$.

A walk that is not closed is called an open walk.

Definition: Cycle.

A closed walk in which no point except the terminal point appear more than one is called a cycle of length n.

A closed walk $v_0, v_1, v_2, \ldots, v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, v_2, \ldots, v_{n-1}$ are distinct is called a cycle of length n.

The graph consisting of cycle of length n is denoted by C_n.

C_3 is called a triangle.

Definition: Trail

A walk is called a trail if all its lines are distinct.

Definition: Path

A walk is called a path if all its points are distinct.

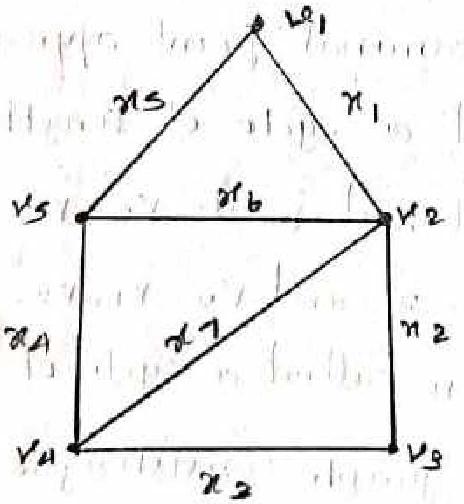
Note:

1) Every path is a trail and a trail need not be a path.

2) The graph consisting of a trail with n points is denoted by P_n.

3) The length of a path in which the number of lines in the path.

Example:



1) v_1, v_2, v_3, v_4, v_5 is a walk.

It is a $v_1 - v_5$ walk.

Initial point of the walk is v_1.

Terminal point of the walk is v_5

The length of the walk is 4.

2) $v_1, v_2, v_3, v_4, v_2, v_1, v_2, v_5$ is a walk

3) v_1, v_2, v_4, v_5, v_2 and $v_1, v_2, v_3, v_4, v_5, v_1$ are closed walk.

4) $v_1, v_2, v_4, v_3, v_2, v_5$ is a trail but not a path.

5) v_1, v_2, v_4, v_5 is a path.

6) v_2, v_3, v_4, v_2 is a cycle.

Theorem 3.1:

In a graph G, any u-v walk contains a u-v path.

Proof.

We prove the result by induction on the length of the walk.

Any walk of length zero or one itself is a path.

Assume the result for all walks of length $< n$.

Prove the result for a walk of length n.

If all the points $u_0, u_1, u_2, \ldots, u_n$ are distinct then this walk itself is a path.

If not, there exist i and j such that $0 \le i < j \le n$ and $u_i = u_j$.

now $u = u_0, u_1, u_2, \ldots, u_i, u_{j+1}, \ldots, u_n = v$ is a $u-v$ walk of length $< n$.

$\therefore$ By induction hypothesis this walk contains $u-v$ path.

Hence, any $u-v$ walk contains a $u-v$ path.

Theorem 3.2

If $\delta \geq k$ then G is a path of length k.

Proof.

Let δ be the minimum degree of the graph G.

Let k be the number of vertices of the graph G.

Let $P = \{v_0, v_1, v_2, \ldots, v_n\}$ be the longest path in G.

Then every vertex adjacent to v_0 lies on P.

Since $\deg v_0 \geq \delta$, the length of $P \geq \delta$ and $\delta \geq k$.

Hence $P_1 = \{v_0, v_1, v_2, \ldots, v_k\}$ is a path of length k in G.

Theorem 3.3

A closed walk of odd length contains a cycle.

Proof

Let $v_0, v_1, v_2, \ldots, v_n = v_0$ be a closed walk of odd length n.

clearly $n \geq 3$

If $n = 3$ then the closed walk of length there is a triangle which is trivially a cycle.

$\therefore$ The result is true for $n = 3$.

Assume the result is true for all walks of length $< n$.

To prove the result for a closed walk $v_0, v_1, v_2, \ldots, v_n = v_0$ of odd length n.

If all the points in this walk are distinct then this walk itself is a cycle.

If not there exist two positive integers i and j such that $i < j$.

$\{i, j\} \neq \{0, n\}$ and $v_i = v_j$, where n is odd.

Now $v_i, v_{i+1}, \ldots, v_j$ and

$v_0, v_1, v_2, \ldots, v_i, v_{j+1}, \ldots, v_n = v_0$

are closed walks contained in the given walk and the sum of their lengths is n.

Since n is odd, at least one of these walks is of odd length.

Hence by induction hypothesis this closed walk contains a cycle.

∴ By the principle of Mathematical induction, the theorem is true for all odd length n.

Solved Problems

1. If A is the adjacency matrix of a graph with $V = \{v_1, v_2, \ldots, v_p\}$ then prove that $n \geq 1$ the $(i, j)^{th}$ entry of A^n is the number of $v_i - v_j$ walks of length n.

Solution

Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_p\}$.

We know that,

The adjacency matrix $A = (a_{ij})_{p \times p}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

We prove the result by induction on n.

When $n = 1$,

The number of $v_i - v_j$ walks of length 1.

$$= \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

$= (a_{ij}) = (i, j)^{th}$ element of A.

$\therefore$ The result is true for $n = 1$.

Assume that the result is true for $n-1$.

Let $A^{n-1} = (a_{ij}^{(n-1)})$.

$\therefore a_{ij}^{(n-1)}$ = the number of $v_i - v_j$ walks of length $(n-1)$.

Now, $A^n = A^{n-1} \cdot A$

$= (a_{ij}^{(n-1)})_{p \times p} (a_{ij})_{p \times p}$

$\therefore (i, j)^{th}$ entry of $A^n = \sum_{i=1}^{p} a_{ik}^{(n-1)} a_{kj} \longrightarrow (1)$

Take

$$a_{ik}^{(n-1)} a_{kj} = \begin{cases} a_{ij}^{(n-1)}, & \text{if } a_{kj} = 1 \text{ i.e. if } v_k \text{ is adjacent to } v_j \\ 0 & \text{if } v_k \text{ is not adjacent to } v_j \end{cases}$$

By induction hypothesis the $(i, j)^{th}$ entry of A^{n-1} is the number of walks of length $n-1$ between v_i and v_k if v_k is adjacent to v_j then the above walk can be made into walks of length n between v_i and v_j.

$\therefore (i, j)^{th}$ entry of A^n is the number of walks of length n between v_i and v_j.

Hence the theorem.

CONNECTEDNESS AND COMPONENTS

Definition: Connectedness

Two points u and v of a graph G are said to be connected if there exists a $u-v$ path.

Definition: Connected

A graph G is said to be connected if there is at least one path between every pair of vertices in G.

A graph G which is not connected is said to be disconnected.

Definition: Component

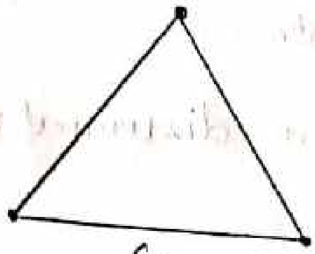
Each of the connected graph is called component.

A graph G is connected iff it has exactly one component.

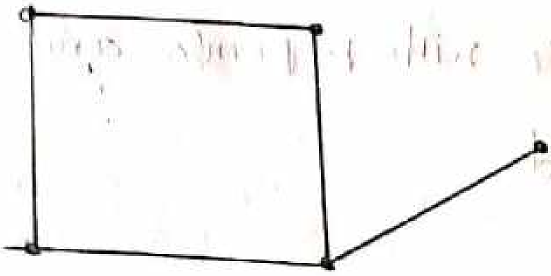
A graph G is disconnected then G has atleast two components.

Example: Connected Graphs.

G_1

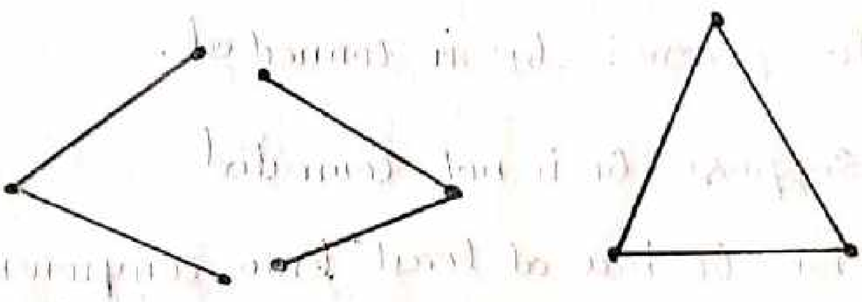


G_2

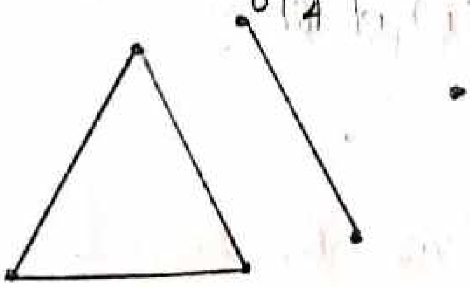


G_3.

Disconnected Graphs.



G_4



G_5

G_1 is a connected graph with one component.

G_2 is a connected graph with one component.

G_3 is a connected graph with one component

G_4 is a disconnected graph with two components.

G_5 is a disconnected graph with three components.

Theorem 3.4

A graph G with p points and $\delta \geq \frac{p-1}{2}$ is connected.

Proof:

Let G be a graph and $\delta \geq \frac{p-1}{2}$ is connected.

To prove: G is connected.

Suppose G is not connected

Then G has at least two components.

Consider $G_1 = (V_1, X_1)$ of G.

Let $v_1 \in V_1$

we have $\delta \geq \frac{p-1}{2}$.

$\therefore \exists$ at least $\frac{p-1}{2}$ points in G_1 which are adjacent to v_1.

$\therefore V_1$ contains at least $\frac{p-1}{2} + 1$ points

i.e. V_1 contains $\frac{p+1}{2}$.

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Also G has at least two components.

$\therefore$ The number of points in $G \geq \frac{p+1}{2} + \frac{p+1}{2}$

i.e. $p \geq p+1$

Which is a contradiction to G has p points.

Hence G is connected.

Theorem 3.5

A graph G is connected iff for any partition of V into subsets V_1 and V_2 there is a line of G joining a points of V_1 to a point of V_2.

Proof.

Assume that G is connected.

Let $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$.

To prove: There is a line of G joining a point of V_1 to point of V_2.

Let $u \in V_1$ and $v \in V_2$.

Since G is connected, there exists a $u-v$ path in G.

Let $u = v_0, v_1, v_2, \ldots, v_n = v$ be a path.

Let i be the least positive integer such that $v_i \in V_2$.

Then $v_{i-1} \in V_1$.

$\therefore$ The line $v_{i-1}v_i$ joins a point of V_1 to a point of V_2.

Hence there is a line of G joining a point

Conversely, assume that there is a line of G joining a point of V_1 to a point of V_2.

To prove: G is connected.

Suppose G is not connected.

Then G contains at least two components say, G_1 and G_2.

Let V_1 be the set of all points of G_1 and V_2 be the set of all points of V_2.

Clearly $V = V_1 \cup V_2$ is a partition of V.

Also there is no line joining any point of V_1 to a point of V_2.

Which is a contradiction to the assumption.

$\therefore$ G is connected

Theorem 3.6

If G is not connected then $\bar{G}$ is connected

Proof.

Suppose G is not connected

Then G has at least two components.

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Let u and v be any two points of G.

If u and v belong to different components of G then they are not adjacent in G.

$\therefore$ They are adjacent in $\bar{G}$.

Hence $\bar{G}$ is connected.

If u and v lie in the same component of G then they are connected in G.

Choose w in the other component.

Then u, w, v is a $u-v$ path in $\bar{G}$.

Hence $\bar{G}$ is connected.

Definition: Distance

For any two points u, v of a graph we define the distance between u and v by

$$d(u,v) = \begin{cases} \text{the length of a shortest } u-v \text{ path if such a path exists} \\ \infty, \text{ otherwise.} \end{cases}$$

Note.

If G is a connected graph the $d(u,v)$ is always a non negative integer.

Hence, d is a metric on the set of points V.

Theorem 3.7: [Necessary and sufficient condition for a graph to be bipartite]

A graph G with at least two points is bipartite iff all its cycles are of even length.

Proof.

Let G be a graph with at least two points is bipartite.

Then V can be partitioned into two subsets V_1 and V_2 such that every line joins a point of V_1 to a point of V_2.

To prove: All its cycles are of even length.

Let $v_0, v_1, v_2, \ldots, v_n = v_0$ be a cycle of length n.

Choose $v_0 \in V_1$

Then $v_2, v_4, v_6, \ldots \in V_1$ and

$v_1, v_3, v_5, \ldots \in V_2$.

Also $v_n = v_0 \in V_1$

$\therefore n$ is even.

Hence all its cycles are of even length.

Conversely, assume that all cycles in G are of even length.

Without loss of generality we assume that G is connected.

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To prove : G is bipartite.

Let $v_1 \in V_1$

Define $V_1 = \{v \in V / d(v, v_1) \text{ is even}\}$

$V_2 = \{v \in V / d(v, v_1) \text{ is odd}\}$

Clearly, $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$.

claim : Every line of G joints a point of V_1 to a point of V_2.

Suppose two points $u, v \in V_1$ are adjacent.

Let P be the shortest $v_1 - u$ path of length m and let Q be the shortest $v_1 - v$ path of length n.

Since $u, v \in V_1$, both m and n are even.

Let u_1 be the last common point of P and Q.

Then the $v_1 - u_1$ path along P and the $v_1 - u_1$ path along Q are both shortest path and hence have the same length, say i.

Now the $u_1 - u$ path along P, the line uv followed by the $v_1 - u_1$ path along Q form a cycle.

Its length is $= (m-i) + 1 + (n-i)$

$= m + n - 2i + 1$

$=$ odd number.

This is a contradiction to our assumption.

Hence no two points of V_1 are adjacent.

Similarly, we can prove that no two points of V_2 are adjacent.

Thus every line joins a point of V_1 to a point of V_2.

Hence G is Bipartite.

Cut point

A cut point of a graph G is a point whose removal increases the number of components.

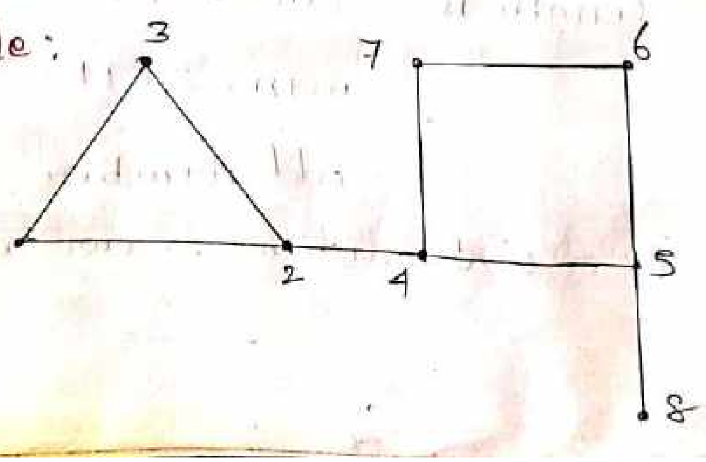
Definition: Bridge

A bridge of a graph G is a line whose removal increases the number of components.

Note

If v is a cut point of a connected graph then $G - \{v\}$ is disconnected.

Example:



2, 4, 5 are cut points.

$\{2, 4\}$ and $\{5, 8\}$ are bridges.

Theorem 3.8

Let v be a point of connected graph G. Then the following statements are equivalent.

(i) v is a cut-point of G.

ii) There exist a partition of $V - \{v\}$ into subsets U and W such that for each $u \in U$ and $w \in W$, the point v is on every $u-w$ path.

iii) There exist two points u and w distinct from v such that v is on every $u-w$ path.

Proof

Let G be a connected graph and v be a point of G.

To prove: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

First, we prove: (i) $\Rightarrow$ (ii).

Assume that v is a cut-point of G.

Then $G - v$ is disconnected.

$\therefore$ $G - v$ contains at least two components.

Let U be the set of points on one component and W be the set of points in the remaining components.

$\therefore V - \{v\} = U \cup W$ and $U \cap W = \phi$.

i.e. There exists a partition of $V - \{v\}$ into subsets U and W.

To prove: each $u \in U$ and $w \in W$, the point v is on every $u-w$ path.

Let $u \in U$ and $w \in W$.

Then u and w lie on different component of $G - v$.

$\therefore$ There is no $u-w$ path in $G - v$.

Hence the point v lies on every $u-w$ path in G.

Secondly, to prove: (ii) $\Rightarrow$ (iii)

This is trivially true.

Thirdly, to prove: (iii) $\Rightarrow$ (i)

Assume that there exists two points u and w distinct from v such that v is on every $u-w$ path.

$\therefore$ There is no $u-w$ path in $G - v$.

$\Rightarrow G - v$ is disconnected.

Hence v is a cut-point of G.

Theorem 2.9:

Let x be a line of a connected graph G.

Then the following statements are equivalent.

(i) x is a bridge of G.

(ii) There exists a partition of V into subsets U and W, such that for every point $u \in U$ and $w \in W$, the line x is on every $u-w$ path.

(iii) There exists two points u and w such that the line x is on every $u-w$ path.

The proof is similar to theorem 2.8 and is left as an exercise.

Theorem 3.10

A line x of a connected graph G is a bridge iff x is not on any cycle of G.

Proof.

Let G be a connected graph $\longrightarrow (1)$

Let the line x be bridge of G

Then $G-x$ is disconnected.

To prove: x is not on any cycle C of G.

Suppose x is on any cycle C of G.

Let w_1 and w_2 be any two points in G.

Since G is connected, there exists a w_1-w_2 path P in G.

If x is not on P, then P itself is a w_1-w_2 path in $G-x$.

$\therefore$ $G-x$ is connected, which is a contradiction to (1).

If x is on P, then replace x by $C-x$, we get a w_1-w_2 walk in $G-x$.

This walk contains a w_1-w_2 path in $G-x$.

$\therefore$ $G-x$ is connected, which is a contradiction to (1).

Conversely, assume that x is not on any cycle of G ——— (2)

To prove: x is a bridge of G.

Suppose x is not a bridge.

Then $G-x$ is connected.

Let $x = uv$.

Then there exists a $u-v$ path in $G-x$.

This path together with the line $x = uv$ forms a cycle containing x.

This is a contradiction to (2)

i.e. to x is not on any cycle of G.

Hence x is a bridge.

Theorem 3.11.

Every non-trivial connected graphs has at least two points which are not cut points.

Proof.

Let G be a non-trivial connected graph. Choose two points u and v such that $d(u,v)$ is maximum.

To prove: u and v are not cut points.

Suppose v is a cut point.

Then $G-v$ is disconnected.

$\therefore$ $G-v$ has at least two components.

Choose a point w in a component that do not contain u.

Then v lies on every $u-w$ path.

$\therefore d(u,w) > d(u,v)$.

This is a contradiction to $d(u,v)$ is maximum.

$\therefore v$ is not a cut point.

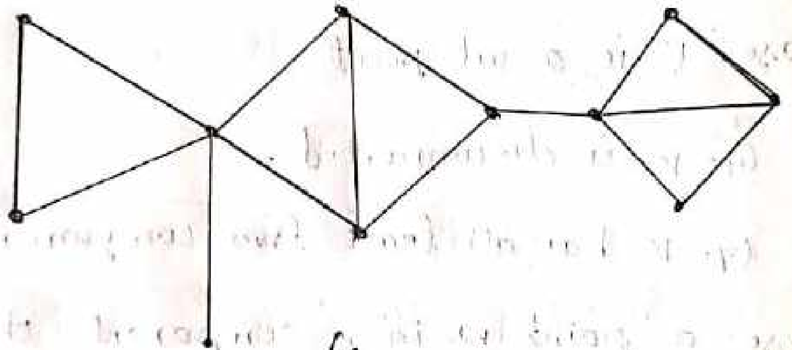
Similarly we can prove that u is not a cut point.

Hence G has at least two points which are not cut points.

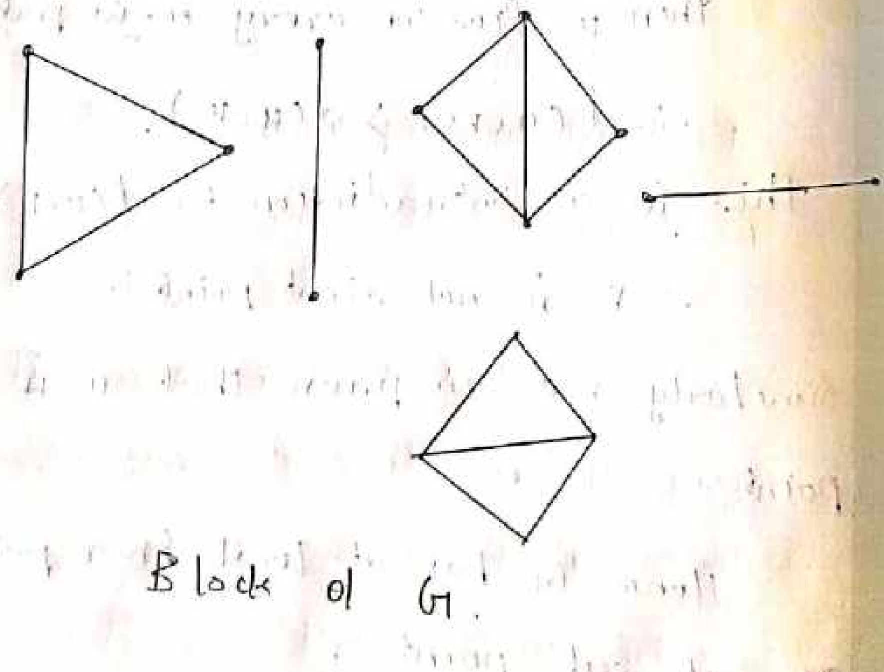
Definition : Blocks

A connected non-trivial graph having no cut point is a block. A block of a graph is a subgraph and which is connected.

Example :



G



Block of G.

Theorem 3.12

Let G be a connected graph with at least three points.

The following statements are equivalent.

(1). G is a block.

(2). Any two points of G lie on a common cycle.

(3). Any point and any line of G lie on a common cycle.

(4). Any two lines of G lie on a common cycle.

Proof:

Let G be a connected graph with at least three points.

(i) To prove: (1) ⟹ (2)

Assume that G is a block.

Then G has no cut points.

To prove that any two points of G lie on a common cycle.

Let u and v be any two points in G.

We prove the result by using induction on $d(u, v)$.

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If $d(u,v) = 1$ then u and v are adjacent and $G \neq K_2$, at least three points.

Also G has no cut points.

$\therefore x = uv$ is not a bridge.

By theorem 3.10, x lies on a common cycle.

Hence the points u and v lie on a common cycle of G.

Assume the result for any two points at distance less than k.

To prove the result for $d(u,v) = k$, where $k \geq 2$.

Consider a $u-v$ path of length k.

Let w be a point that precedes v on this path.

Then $d(u,w) = k-1$

By induction hypothesis, the points u and w lie on a common cycle C of G.

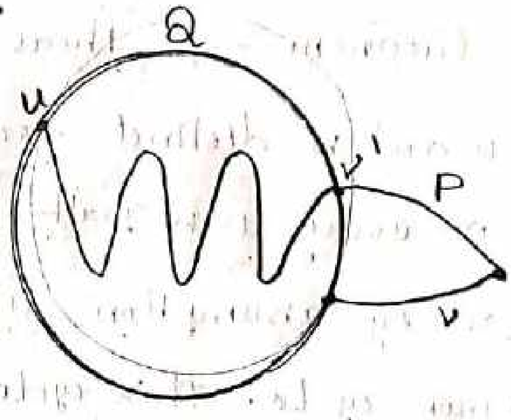
Since G is a block, w is not a cut point of G.

$\therefore G - w$ is connected.

Hence, there exists a $u-v$ path not containing w.

Let v' be the last point common to P and C.

Since u is common to P and C, such a v' exists



Let Q denote the $u-v'$ path along the cycle not containing the point w.

Now, $u-v'$ path along Q, $v'-v$ path along P, the line vw and $w-u$ path along C form a cycle.

This cycle contains both u and v.

Hence by induction, any two points of G lie on a common cycle.

(ii). To prove : (2) $\Rightarrow$ (1).

Assume that any two points of G lie on a common cycle.

To prove that G is block.

Since G is a connected non-trivial graph it is enough to prove that G has no cut point.

Suppose v is a cut point.

By theorem 3.8, there exists two points u and w distinct from v such that v lies on every u-w path.

Also by assumption, u and w lie on a common cycle. this cycle determines two u-w paths and atleast one of these paths does not contain v.

this is a contradiction, since v lies on every u-w path.

∴ v is a cut point.

Hence G is a block.

∴ (1) ⇔ (2)

(ii) To prove : (2) ⇒ (3)

Assume that any two points of G lie on a common cycle of G.

To prove that any point and any line

lie on a common cycle.

Let u be a point and vw be a line of G.

By assumption u and v lie on a common cycle C.

If w lies on C, then the point u and the line vw lie on a common cycle.

If w is not on C, let C' be a cycle containing u and w.

This cycle determines two $w-u$ paths and at least one of them does not contain v.

Denote this path by P.

Let u' be the first point common to P and C.

Now, the line vw, the $w-u'$ path along P, $u'-v$ path in C containing u form a cycle. This cycle contains the point u and the line vw.

(iv) To prove : $(3) \Rightarrow (2)$ is trivial

$\therefore (2) \Leftrightarrow (3)$.

(v) To prove : $(3) \Rightarrow (4)$ is trivial

Assume that any point and any line lie on a common cycle.

To prove that any two lines of G lie on a common cycle.

Let uu_1 and vw be two lines.

By assumption, the point u and the line vw lie on a common cycle C.

Also the point u_1 and the line vw lie on the common cycle c'.

Now the line uu_1, u_1w path along C', the line vw and $v-w$ path along C form a cycle.

This cycle contains the lines uu_1 and vw.

Hence any two lines of G lie on a common cycle.

(vi) To prove: $(4) \Rightarrow (3)$ is trivial.

$\therefore (3) \Leftrightarrow (4)$.

Hence, the statements (1), (2), (3) and (4) are equivalent for any connected graph with at least three points.

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UNIT - V

TREES

The concept of a tree was discovered by Cayley in the year 1857.

Definition: Acyclic Graph

A graph which contains no cycles is called an acyclic graph.

Definition: Tree

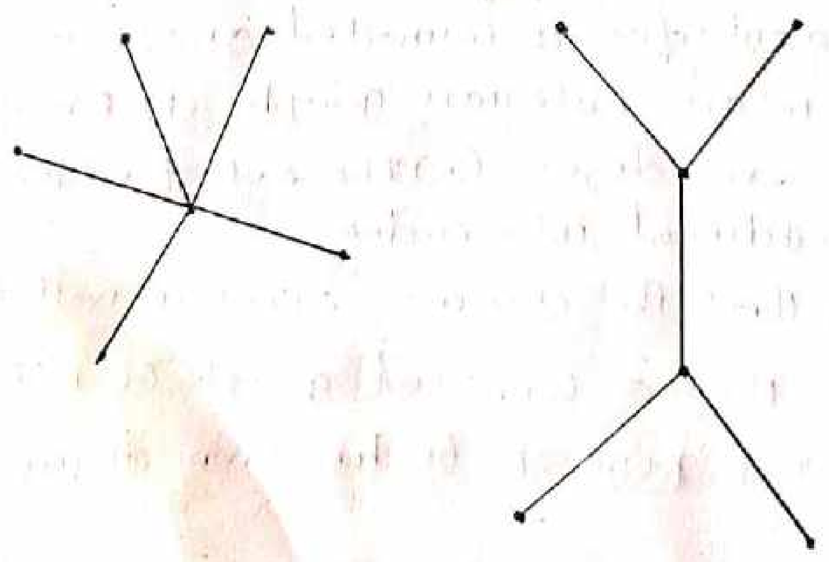
A connected acyclic graph is called a tree.

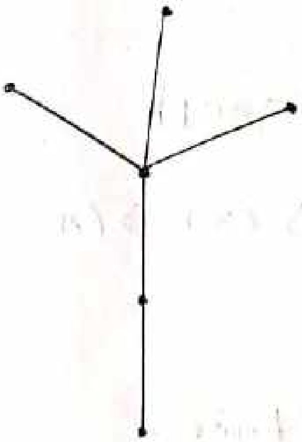
Note:

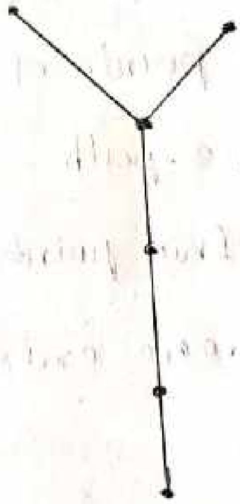
* Any graph without cycles is also called a forest.

* Components of a forest are trees.

Example:







CHARACTERISATION OF TREES

Theorem 5.1

Let G be a (p, q) graph. The following statements are equivalent.

1. G is a tree.
2. Every two points of G are joined by a unique path.
3. G is connected and $p = q + 1$
4. G is acyclic and $p = q + 1$.

Proof.

Let G be a (p, q) graph.

To prove : $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$,

(i) To prove : $(1) \Rightarrow (2)$

Assume that G is a tree.

$\therefore$ G is a connected acyclic graph.

To prove that any two points of G are connected by a unique path.

Let u and v be any two points of G.

Since G is connected there exists a $u-v$ path in G.

Suppose there exists two distinct $u-v$ paths.

$P_1 : u = v_0, v_1, \ldots, v_n = v$ and

$P_2 : u = w_0, w_1, \ldots, w_m = v$.

Let i be the least positive integer such that $1 \leq i < m$ and $w_i \notin P_1$.

$\therefore w_{i-1} \in P_1 \cap P_2$.

Let j be the least positive integer such that $i < j \leq m$ and $w_j \in P_1$.

Then the $w_{i-1} - w_j$ path along P_2

followed by the $w_j - w_{i-1}$ path along P_1 form a cycle.

This is contradiction to G is acyclic.

Hence every two points of G are joined by a unique path.

(ii) To prove: $(2) \Rightarrow (3)$.

Assume that every two points of G are joined by a unique path.

To prove that G is connected and $p = q + 1$.

Clearly, G is connected.

To prove: $p = q + 1$

Let us prove the result by using induction on p.

When $p = 1$ or $p = 2$; $q = 0$ or $q = 1$

$\therefore$ The result is true when $p = 1$ or $p = 2$.

Assume the result for all graphs with less than p points.

To prove the result for a graph G with p points

Let u and v be any two points of G.

Then by the assumption there exists a unique $u-v$ path in G.

Consider any line x on the path.

Then $G-x$ is a disconnected graph with exactly two components G_1 and G_2,

Let G_1 be a (p_1, q_1) graph and G_2 be a (p_2, q_2) graph.

Then $p_1 + p_2 = p$ and $q_1 + q_2 = q - 1$.

Clearly, p_1 and $p_2 < p$.

$\therefore$ By the induction assumption,

$p_1 = q_1 + 1$ and $p_2 = q_2 + 1$.

Now, $p = p_1 + p_2$

$$= q_1 + q_2 + 2$$

$$= q - 1 + 2$$

$$= q + 1.$$

Hence G is connected and $p = q + 1$

(iii). To prove: $(3) \Rightarrow (4)$.

Assume that G is connected and $p = q + 1$

To prove that G is acyclic and $p=q+1$.

Suppose that G contains a cycle of length n.

There are n points and n lines on this cycle.

Fix a point u on the cycle. Consider any one of the remaining $p-n$ points not on the cycle, say v.

Since G is connected we can find a shortest $u-v$ path in G.

Consider the line on this shortest path incident with v.

Then $p-n$ lines thus obtained are all distinct.

$\therefore\ q \geq (p-n)+n$

$\Rightarrow q \geq p$.

This is a contradiction to $p=q+1$.

Hence G is acyclic and $p=q+1$.

(iv) To prove: $(4) \Rightarrow (1)$.

Assume that G is acyclic and $p=q+1$.

To prove that G is a tree.

It is enough to prove that G is connected.

Suppose G is not connected. Then G has more than one component.

Let $G_1, G_2, \dots, G_k$, $k \geq 2$ be the components of G.

Since G is acyclic, each of these components is a tree.

$\therefore p_i = q_i + 1$ for $i = 1, 2, \dots, k$, where G_i is a (p_i, q_i) graph.

$$\therefore \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i + 1$$

$\Rightarrow p = q + k \geq q + 2$, since $p = q + 1$

$\therefore G$ is connected.

Hence G is a tree.

Corollary:

Every non-trivial tree G has atleast two vertices of degree 1.

Proof.

Since G is non-trivial, $d(v) \geq 1$ for all points v.

Also G is a tree.

$\therefore p = q + 1$.

Now, $\sum d(v) = 2q = 2(p-1) = 2p - 2$.

$\Rightarrow d(v) = 1$ for at least two vertices.

Theorem 5.2:

Every connected graph has a spanning tree.

Proof.

Let G be a connected graph.

Let T be a minimal connected spanning subgraph of G.

To prove that T is a tree.

By the definition of T, for any line x of T, $T - x$ is disconnected.

$\therefore$ x is a bridge of T.

We know that, "A line x of a connected graph G is a bridge iff x is not on any cycle of G"

$\therefore$ x is not on any cycle of T.

$\therefore$ T is acyclic.

Further, T is connected.

$\therefore$ T is a tree.

Hence T is a spanning tree of G.

Corollary:

Let G be a (p, q) connected graph. Then $q \geq p-1$

Proof.

Let G be a (p, q) connected graph.

We know that, "Every connected graph has a spanning tree".

$\therefore$ G has a spanning tree T.

$\therefore$ T has p points and $p-1$ lines.

Hence $q \geq p-1$.

Theorem 5.3

Let T be a spanning tree of a connected graph G. Let $x = uv$ be an edge of G not in T. Then $T+x$ contains a unique cycle.

Proof.

Let T be a spanning tree of a connected graph G.

Also $x = uv$ be an edge of G not in T.

We have T is acyclic.

$\therefore$ Every cycle in $T+x$ must contain the edge x.

Hence there exists a one to one correspondence between the cycles in $T+x$ and $u-v$ path in tree T.

We know that, In a tree there is a unique $u-v$ path.

Hence there is a unique cycle in $T+x$.

CENTRE OF A TREE

Definition: Eccentricity

Let v be a point in a connected graph G. The eccentricity $e(v)$ is defined by $e(v) = \max\{d(u,v) / u \in V(G)\}$.

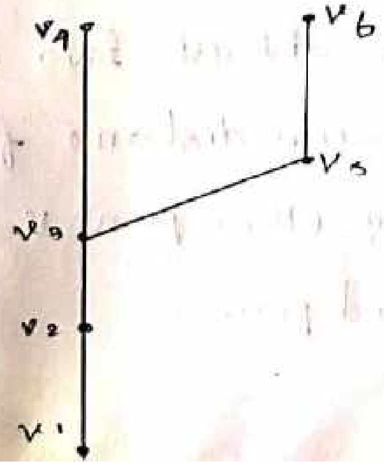
Definition: Radius

The radius $r(G)$ is defined by

$r(G) = \min\{e(v) / v \in V(G)\}$.

v is called a central point if $e(v) = r(G)$ and the set of all central points is called the centre of G.

Example.



The eccentricity

$e(v_1) = 4$; $e(v_4) = 3$

$e(v_2) = 3$; $e(v_5) = 3$

$e(v_3) = 2$; $e(v_6) = 4$

The radius $r(G) = \min \{e(v) / v \in V(G)\}$

$= 2$

Centre of $G = \{v / e(v) = r(G)\}$

$= v_3$.

Theorem: 5 - 4

Every tree has a centre consisting of either one point or two adjacent points.

Proof.

The result is obvious for the trees K_1 and K_2.

Let T be any tree with $p \geq 2$ points.

Then T has atleast two end points and the maximum distance from a given point u to any other point v occurs only when v is an end point.

Now delete all the end points from T.

The resulting graph T' is also a tree and also the eccentricity of each point in T' is exactly one less than the eccentricity of that vertex in T.

∴ T and T' have the same centre.

The process of removing end points is repeated.

Finally we get a successive trees having the same centre as T.

Hence we obtain a tree which is either K_1 or K_2.

∴ The centre of T consists of either one point or two adjacent points.